Online Supplement to

Portfolio Choice with Competing Precautionary and Accumulation Goals

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S.1 Viscosity Proofs

S.1.1 Deterministic Deadline Goal

As in Section 3, for $K > 0$ we write

$$\Pi_K = [-K, K]^n, \qquad H^K(w, p, q) := \sup_{\boldsymbol{\pi} \in \Pi_K} \left\{ wp\, \boldsymbol{\pi}^\top \boldsymbol{\theta} + \frac{1}{2} w^2 q\, \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \right\}.$$

We write the HJB equation appearing in (20) either as

$$u_t + H^K(w, u_w, u_{ww}) = 0 \tag{S.1}$$

or, equivalently, in terms of

$$\mathcal{F}_D^K(t, w, z, a, p, q) := -a - H^K(w, p, q).$$

Here z is the value variable. It is included only to match standard viscosity notation since the fixed-deadline equation has no zero-order term.

A viscosity subsolution of $\mathcal{F}_D^K = 0$ satisfies $\mathcal{F}_D^K \leq 0$. Thus, if a smooth test function ϕ touches a subsolution from above at (t, w), then

$$-\phi_t(t, w) - H^K(w, \phi_w(t, w), \phi_{ww}(t, w)) \leq 0,$$

equivalently $\phi_t + H^K \geq 0$. A viscosity supersolution satisfies the reverse inequality. This is the sign convention used throughout the proof below. We begin with a preliminary result about H^K.

Lemma S.1.1. *For every $K > 0$, the Hamiltonian H^K is finite and continuous on $\mathbb{R}^3$. Moreover, on every compact subset of $\mathbb{R}^3$, H^K is Lipschitz. Finally, if $q_1 \leq q_2$, then*

$$H^K(w, p, q_1) \leq H^K(w, p, q_2) \qquad \text{for all } (w, p) \in \mathbb{R}^2.$$

Consequently, $\mathcal{F}_D^K$ is continuous, proper, and degenerate elliptic.

Proof. For $\boldsymbol{\pi} \in \Pi_K$, define

$$g^K(w, p, q; \boldsymbol{\pi}) := w\, p\, \boldsymbol{\pi}^\top \boldsymbol{\theta} + \tfrac{1}{2}\, w^2\, q\, \boldsymbol{\pi}^\top \boldsymbol{\Sigma}\, \boldsymbol{\pi}.$$

Fix a compact set $C \subset \mathbb{R}^3$. Because Π_K is compact, there is a constant $M_K < \infty$ such that $\|\boldsymbol{\pi}\| \leq M_K$ for every $\boldsymbol{\pi} \in \Pi_K$. Since g^K is polynomial in $(w, p, q, \boldsymbol{\pi})$, there exists $L_{C,K} < \infty$ with

$$|g^K(\xi; \boldsymbol{\pi}) - g^K(\eta; \boldsymbol{\pi})| \leq L_{C,K}\, \|\xi - \eta\| \qquad \text{for all } \xi, \eta \in C,\ \boldsymbol{\pi} \in \Pi_K.$$

Taking the supremum over $\boldsymbol{\pi} \in \Pi_K$ yields

$$|H^K(\xi) - H^K(\eta)| \leq L_{C,K}\, \|\xi - \eta\|, \qquad \xi, \eta \in C,$$

so H^K is continuous and locally Lipschitz.

If $q_1 \leq q_2$, then for every $\boldsymbol{\pi} \in \Pi_K$,

$$g^K(w, p, q_1; \boldsymbol{\pi}) \leq g^K(w, p, q_2; \boldsymbol{\pi}),$$

because $\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\, \boldsymbol{\pi} \geq 0$. Taking suprema gives $H^K(w, p, q_1) \leq H^K(w, p, q_2)$. Therefore $\mathcal{F}_D^K$ is nonincreasing in q, i.e. degenerate elliptic. It is proper because it does not depend on the value variable (therefore nondecreasing), and it is continuous by the continuity of H^K. □

We note that since the control set Π_K is compact, the drift and diffusion coefficients for the wealth

$$b(w, \boldsymbol{\pi}) = w \boldsymbol{\pi}^\top \boldsymbol{\theta}, \qquad \sigma_W(w, \boldsymbol{\pi}) = w \boldsymbol{\pi}^\top \boldsymbol{\sigma}$$

are locally Lipschitz in w with constants uniform in $\boldsymbol{\pi} \in \Pi_K$, and satisfy a uniform linear-growth bound. Hence, for every admissible control, the wealth equation has a unique strong solution and the problem is well-posed.

Lemma S.1.2. *The value function $V^{D,K}(t, w)$ is jointly continuous on $[0, T] \times [0, \infty)$.*

Proof. Since the function F_G is continuous and constant outside the compact interval $[0, b_G]$, it is uniformly continuous on $\mathbb{R}$. Choose Lipschitz functions $F_G^m : \mathbb{R} \to [0, 1]$ such that

$$F_G^m(x) = 0 \quad \text{for } x \leq 0, \qquad F_G^m(x) = 1 \quad \text{for } x \geq b_G, \qquad \|F_G^m - F_G\|_\infty \to 0.$$

For instance, one may take the piecewise-linear interpolations of F_G on partitions of $[0, b_G]$ whose mesh tends to zero, and extend them constantly outside $[0, b_G]$.

For each m, define the auxiliary value function

$$V_m(t,w) := \sup_{\boldsymbol{\pi}\in\mathcal{A}_{t,T}^{K,\mathrm{pre}}} \mathbb{E}\Big[F_G^m\Big(\widetilde{W}_T^{t,w;\boldsymbol{\pi}}\Big)\Big], \qquad (t,w)\in[0,T]\times\mathbb{R}.$$

Observe that we have extended the domain to all of $\mathbb{R}$ but this does not pose a problem since the wealth SDE is still well-posed and clearly $V_m(t,w) = 0$ if $w \le 0$. For fixed m, this is a finite-horizon stochastic control problem in Mayer form with bounded control set Π_K, Lipschitz terminal payoff F_G^m, and coefficients that have linear growth in (w, π) and are Lipschitz in the state uniformly over controls. Therefore [Touzi, 2013, Proposition 3.7] applies and gives $V_m \in C([0,T]\times\mathbb{R})$. More precisely, V_m is Lipschitz in the wealth variable, uniformly in time, and $1/2$-HÃ¶lder continuous in time on compact wealth intervals.

We now compare V_m with $V^{D,K}$. For every $(t,w)\in[0,T]\times[0,\infty)$,

$$\begin{aligned}\left|V_m(t,w) - V^{D,K}(t,w)\right| &\le \sup_{\boldsymbol{\pi}\in\mathcal{A}_{t,T}^{K,\mathrm{pre}}} \mathbb{E}\Big[\Big|F_G^m\Big(\widetilde{W}_T^{t,w;\boldsymbol{\pi}}\Big) - F_G\Big(\widetilde{W}_T^{t,w;\boldsymbol{\pi}}\Big)\Big|\Big] \\ &\le \|F_G^m - F_G\|_\infty.\end{aligned}$$

Thus $V_m \to V^{D,K}$ uniformly on $[0,T]\times[0,\infty)$. Since each V_m is continuous, the uniform limit $V^{D,K}$ is continuous on $[0,T]\times[0,\infty)$. ☐

Our next result gives the usual comparison principle for viscosity solutions. For an open domain $Q \subset \mathbb{R}^2$, we write $USC(\overline{Q})$ and $LSC(\overline{Q})$ for the classes of real-valued functions on $\overline{Q}$ that are upper and lower semicontinuous relative to $\overline{Q}$, respectively.

Lemma S.1.3 (Comparison). *Let $Q := (0,T)\times(0,b_G)$, and let $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$ be bounded. Suppose that u is a viscosity subsolution and v is a viscosity supersolution of*

$$-u_t - H^K(w, u_w, u_{ww}) = 0 \qquad \textit{in } Q.$$

Assume further that the parabolic boundary inequalities

$$u(T,w) \le F_G(w) \le v(T,w), \qquad 0 \le w \le b_G,$$

and

$$u(t,0) \le 0 \le v(t,0), \qquad u(t,b_G) \le 1 \le v(t,b_G), \qquad 0 \le t \le T,$$

hold. Then $u \le v$ on $(0,T]\times[0,b_G]$.

Proof. We reduce the statement to the Cauchy–Dirichlet comparison theorem of [Crandall et al., 1992, Theorem 8.2]. Set $s := T - t$, $h(w) := \frac{w}{b_G}$, and define

$$U(s,w) := u(T-s,w) - h(w), \qquad V(s,w) := v(T-s,w) - h(w), \qquad (s,w)\in[0,T]\times[0,b_G].$$

Then $U \in USC([0,T] \times [0,b_G])$ and $V \in LSC([0,T] \times [0,b_G])$. The time reversal turns the terminal condition into an initial condition, and the subtraction of h makes it so

$$U(0,w) \le F_G(w) - h(w) \le V(0,w), \qquad 0 \le w \le b_G,$$

and

$$U(s,0) \le 0 \le V(s,0), \qquad U(s,b_G) \le 0 \le V(s,b_G), \qquad 0 \le s \le T.$$

We now identify the transformed equation. If ψ is a smooth test function for U, then the corresponding test function for u is

$$\phi(t,w) := \psi(T-t,w) + h(w).$$

Thus

$$\phi_t(t,w) = -\psi_s(T-t,w), \qquad \phi_w(t,w) = \psi_w(T-t,w) + h'(w), \qquad \phi_{ww}(t,w) = \psi_{ww}(T-t,w) + h''(w).$$

Since $h'(w) = 1/b_G$ and $h''(w) = 0$, the viscosity inequalities for u and v transform into the viscosity inequalities for U and V associated with

$$z_s + F_h(w, z_w, z_{ww}) = 0 \qquad \text{in } (0,T) \times (0,b_G),$$

where

$$F_h(w,p,q) := -H^K\left(w, p + \frac{1}{b_G}, q\right).$$

Hence U is a viscosity subsolution and V is a viscosity supersolution of this Cauchy–Dirichlet problem with continuous initial data $F_G - h$. This is precisely the problem in [Crandall et al., 1992, Equation (8.4)].

It remains to verify the structural assumptions needed for [Crandall et al., 1992, Theorem 8.2]. By Lemma S.1.1 we conclude that the operator F_h is continuous and proper. Continuity follows from the continuity of H^K. Properness holds because F_h is independent of the value variable and is degenerate elliptic: if $q_1 \le q_2$, then $H^K(w,p,q_1) \le H^K(w,p,q_2)$, and therefore $F_h(w,p,q_1) \ge F_h(w,p,q_2)$.

Lastly, the structural condition [Crandall et al., 1992, Equation (3.14)] holds by a direct application of [Crandall et al., 1992, Example 3.6]. Indeed, for each $\boldsymbol{\pi} \in \Pi_K$ define

$$b^{\boldsymbol{\pi}}(w) := -w\boldsymbol{\pi}^\top \boldsymbol{\theta}, \qquad Z^{\boldsymbol{\pi}}(w) := \frac{w}{\sqrt{2}}\left(\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}\right)^{1/2}, \qquad f^{\boldsymbol{\pi}}(w) := \frac{w}{b_G}\boldsymbol{\pi}^\top \boldsymbol{\theta}.$$

Then, identifying $Z^{\boldsymbol{\pi}}(w)$ with a 1×1 matrix,

$$F_h(w,p,q) = \inf_{\boldsymbol{\pi} \in \Pi_K} \left\{ b^{\boldsymbol{\pi}}(w)p - \operatorname{tr}\left(Z^{\boldsymbol{\pi}}(w)^\top Z^{\boldsymbol{\pi}}(w) q\right) - f^{\boldsymbol{\pi}}(w) \right\}.$$

For each fixed $\boldsymbol{\pi}$, this is a sum of the first-order, second-order, and continuous spatial terms covered by Example 3.6. Moreover, because Π_K is compact, the one-sided Lipschitz constants for $b^{\boldsymbol{\pi}}$, the Lipschitz constants for $Z^{\boldsymbol{\pi}}$, and the moduli of continuity of $f^{\boldsymbol{\pi}}$ can be chosen uniformly over $\boldsymbol{\pi} \in \Pi_K$. The final stability statement in [Crandall et al., 1992, Example 3.6] therefore implies that the infimum over $\boldsymbol{\pi} \in \Pi_K$ satisfies the same structural condition.

The Cauchy–Dirichlet comparison theorem [Crandall et al., 1992, Theorem 8.2] yields

$$U(s,w) \le V(s,w) \qquad \text{for all } (s,w) \in [0,T) \times [0,b_G].$$

Since $U - V = u(T-s,w) - v(T-s,w)$, returning to the original variables gives $u \le v$ on $(0,T] \times [0,b_G]$. □

With these preliminaries, we are ready to complete the proof of Proposition 4.1.

Proof of Proposition 4.1. We divide the proof into parts

Step 1: (Regularity and Boundary Values) The value function is bounded because $0 \le F_G \le 1$ and it is continuous by Lemma S.1.2. The boundary values are immediate. If $w = 0$, then wealth remains zero and $V^{D,K}(t,0) = F_G(0) = 0$. If $w = b_G$, the household can choose $\boldsymbol{\pi} \equiv 0$ and lock in terminal wealth b_G, so $V^{D,K}(t,b_G) = 1$. At $t = T$, no time remains to trade, and therefore $V^{D,K}(T,w) = F_G(w)$.

Step 2: (Viscosity Solution) We note as in Lemma S.1.2 that the definition of $V^{D,K}$ readily extends to $[0,T] \times \mathbb{R}$ with the same properties. We note that the present bounded-control Mayer problem falls within the stochastic-control framework of [Touzi, 2013, Section 7]. Since $V^{D,K}$ is bounded and H^K is finite and continuous, we may apply [Touzi, 2013, Theorem 7.4] to obtain that $V^{D,K}$ is a viscosity solution of (20) on $[0,T) \times \mathbb{R}$ and hence also on $[0,T) \times (0,b_G)$.

Step 3: (Uniqueness) Let u and v be two bounded continuous viscosity solutions of

$$-u_t - H^K(w, u_w, u_{ww}) = 0 \qquad \text{in } (0,T) \times (0,b_G),$$

with the stated boundary data. Applying Lemma S.1.3 to u as subsolution and v as supersolution gives $u \le v$. Reversing the roles of u and v gives $v \le u$. Hence $u = v$ on $(0,T] \times [0,b_G]$. This equality extends to 0 by continuity. □

S.1.2 Random Deadline Goal

Define

$$\mathcal{F}_R^K(w,z,p,q) := \lambda z - H^K(w,p,q) - \lambda F_R(w),$$

so that (25) is

$$\mathcal{F}_R^K\big(w, V^{R,K}(w), (V^{R,K})'(w), (V^{R,K})''(w)\big) = 0.$$

Because the same clipped Hamiltonian H^K appears here as in the fixed-deadline problem, Lemma S.1.1 implies that H^K is finite and continuous, locally Lipschitz on compact sets, and nondecreasing in its second-derivative argument. Since F_R is continuous, $\mathcal{F}_R^K$ is continuous. Moreover, $\mathcal{F}_R^K$ is strictly increasing in the value variable z, with slope $\lambda > 0$, and is nonincreasing in the second-derivative argument q. Hence $\mathcal{F}_R^K$ is proper and degenerate elliptic.

Our first supporting result proves the continuity of the value function $V^{R,K}$.

Lemma S.1.4. *The value function $V^{R,K}$ is continuous on $[0,\infty)$.*

Proof. Since F_R is continuous and constant outside $[0,b_R]$, it is uniformly continuous on $\mathbb{R}$. Let ω_R be a modulus of continuity for F_R. Fix $N>0$ and define the truncated value

$$V_N^{R,K}(w) := \sup_{\boldsymbol{\pi}\in\mathcal{A}_{0,\infty}^{K,\mathrm{pre}}} \mathbb{E}\left[\int_0^N \lambda e^{-\lambda s} F_R\Big(\widetilde{W}_s^{0,w;\boldsymbol{\pi}}\Big)\, ds\right].$$

The tail is uniformly bounded by

$$0 \le V^{R,K}(w) - V_N^{R,K}(w) \le \int_N^\infty \lambda e^{-\lambda s}\, ds = e^{-\lambda N}.$$

Hence, for $w,w'\ge 0$,

$$|V^{R,K}(w) - V^{R,K}(w')| \le |V_N^{R,K}(w) - V_N^{R,K}(w')| + 2e^{-\lambda N}.$$

We now prove continuity of $V_N^{R,K}$. Couple the wealth processes started from w and w' using the same Brownian motion and the same control $\boldsymbol{\pi}$. Since Π_K is bounded, the coefficients

$$b(x,\boldsymbol{\pi}) = x\boldsymbol{\pi}^\top\boldsymbol{\theta}, \qquad \sigma_W(x,\boldsymbol{\pi}) = x\boldsymbol{\pi}^\top\boldsymbol{\sigma}$$

are globally Lipschitz in x, uniformly over $\boldsymbol{\pi}\in\Pi_K$, and have linear growth. Therefore the finite-horizon flow estimate of [Touzi, 2013, Theorem 2.4] gives, for each $N<\infty$, a constant $C_{N,K}<\infty$ such that

$$\sup_{\boldsymbol{\pi}\in\mathcal{A}_{0,\infty}^{K,\mathrm{pre}}} \mathbb{E}\left[\sup_{0\le s\le N}\left|\widetilde{W}_s^{0,w;\boldsymbol{\pi}} - \widetilde{W}_s^{0,w';\boldsymbol{\pi}}\right|^2\right] \le C_{N,K}|w-w'|^2.$$

Consequently, for every $\eta>0$,

$$\begin{aligned}
|V_N^{R,K}(w) - V_N^{R,K}(w')| &\le \sup_{\boldsymbol{\pi}} \mathbb{E}\left[\int_0^N \lambda e^{-\lambda s}\left|F_R\Big(\widetilde{W}_s^{0,w;\boldsymbol{\pi}}\Big) - F_R\Big(\widetilde{W}_s^{0,w';\boldsymbol{\pi}}\Big)\right| ds\right] \\
&\le \omega_R(\eta) + 2\sup_{\boldsymbol{\pi}} \mathbb{P}\left(\sup_{0\le s\le N}\left|\widetilde{W}_s^{0,w;\boldsymbol{\pi}} - \widetilde{W}_s^{0,w';\boldsymbol{\pi}}\right| > \eta\right) \\
&\le \omega_R(\eta) + \frac{2C_{N,K}}{\eta^2}|w-w'|^2.
\end{aligned}$$

Letting $w'\to w$ gives

$$\limsup_{w'\to w} |V_N^{R,K}(w) - V_N^{R,K}(w')| \le \omega_R(\eta).$$

Then $\eta\downarrow 0$ proves continuity of $V_N^{R,K}$. Finally,

$$\limsup_{w'\to w} |V^{R,K}(w) - V^{R,K}(w')| \le 2e^{-\lambda N},$$

and letting $N \to \infty$ proves continuity of $V^{R,K}$. $\square$

Next, we prove the comparison principle for viscosity solutions of this elliptic problem.

Lemma S.1.5 (Comparison)**.** *Let $\Omega_R := (0, b_R)$. Suppose $u \in USC(\overline{\Omega}_R)$ and $v \in LSC(\overline{\Omega}_R)$ are bounded, u is a viscosity subsolution and v is a viscosity supersolution of*

$$\mathcal{F}_R^K(w, u, u', u'') = 0 \qquad \textit{in } \Omega_R,$$

and

$$u(0) \le v(0), \qquad u(b_R) \le v(b_R).$$

Then $u \le v$ on $\overline{\Omega}_R$.

Proof. We verify the hypotheses of [Crandall et al., 1992, Theorem 3.3]. The domain Ω_R is bounded, and as noted above $\mathcal{F}_R^K$ is continuous and proper. Moreover, for $r > s$,

$$\mathcal{F}_R^K(w, r, p, q) - \mathcal{F}_R^K(w, s, p, q) = \lambda(r - s),$$

so the condition in [Crandall et al., 1992, Equation (3.13)] holds with $\gamma = \lambda$.

It remains only to verify the structural condition [Crandall et al., 1992, Equation (3.14)]. We write

$$\mathcal{F}_R^K(w, r, p, q) = \lambda r + \inf_{\boldsymbol{\pi} \in \Pi_K} \left\{ -wp\, \boldsymbol{\pi}^\top \boldsymbol{\theta} - \frac{1}{2} w^2 q\, \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} - \lambda F_R(w) \right\}.$$

For each $\boldsymbol{\pi} \in \Pi_K$, set

$$b^{\boldsymbol{\pi}}(w) := -w \boldsymbol{\pi}^\top \boldsymbol{\theta}, \qquad Z^{\boldsymbol{\pi}}(w) := \frac{w}{\sqrt{2}} \left(\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}\right)^{1/2}.$$

Then the controlled part has the form

$$b^{\boldsymbol{\pi}}(w) p - \operatorname{tr}\left(Z^{\boldsymbol{\pi}}(w)^\top Z^{\boldsymbol{\pi}}(w) q\right) - \lambda F_R(w).$$

The first-order coefficients $b^{\boldsymbol{\pi}}$ are one-sided Lipschitz in w uniformly over $\boldsymbol{\pi} \in \Pi_K$, the diffusion factors $Z^{\boldsymbol{\pi}}$ are Lipschitz in w uniformly over $\boldsymbol{\pi} \in \Pi_K$, and the spatial term $\lambda F_R(w)$ is continuous on the compact interval $[0, b_R]$. Therefore [Crandall et al., 1992, Example 3.6] gives the structural condition with a common modulus, and this condition is preserved under the infimum over the compact control set Π_K.

The comparison theorem now yields $u \le v$ in Ω_R. The boundary inequality gives the result on $\overline{\Omega}_R$. $\square$

We now turn to the proof of Proposition 4.2.

Proof of Proposition 4.2. We once again divide the proof into parts

Step 1: (Regularity and Boundary Values) The value is bounded between 0 and 1 since $0 \le F_R \le 1$ and continuity follows from Lemma S.1.4. The boundary values are also immediate as before with $V^{D,K}$. If

$w = 0$, wealth remains zero and $F_R(0) = 0$, so $V^{R,K}(0) = 0$. If $w \geq b_R$, the household can choose $\boldsymbol{\pi} \equiv 0$, keeping wealth at least b_R forever, and hence $V^{R,K}(w) = 1$.

Step 2: (Viscosity Solution) To see that the value function $V^{R,K}$ is a viscosity solution of

$$\lambda u(w) - H^K\big(w, u'(w), u''(w)\big) - \lambda F_R(w) = 0 \qquad \text{on } (0, b_R)$$

we view the random-deadline problem as an infinite-horizon discounted stochastic control problem, in the framework of [Pham, 2009, Section 4.3]. To match the Euclidean-state formulation in [Pham, 2009, Section 4.3], we use the harmless extension of the coefficients and payoff to all $w \in \mathbb{R}$. The state dynamics are autonomous:

$$d\widetilde{W}_t = \widetilde{W}_t \left(\boldsymbol{\pi}_t^\top \boldsymbol{\theta}\, dt + \boldsymbol{\pi}_t^\top \boldsymbol{\sigma}\, dB_t \right), \qquad \boldsymbol{\pi}_t \in \Pi_K$$

and the control set is compact. Starting from $w > 0$, the solution remains strictly positive. The coefficients are continuous, locally Lipschitz in the state uniformly over $\boldsymbol{\pi} \in \Pi_K$, and satisfy a linear-growth bound. The running reward is

$$f(w, \boldsymbol{\pi}) := \lambda F_R(w),$$

which is bounded and continuous, and the discount rate is $\lambda > 0$.

In the notation of Pham [2009], the Hamiltonian is

$$\widehat{H}^K(w,p,q) := \sup_{\boldsymbol{\pi} \in \Pi_K} \left\{ wp\, \boldsymbol{\pi}^\top \boldsymbol{\theta} + \frac{1}{2} w^2 q\, \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} + \lambda F_R(w) \right\} = H^K(w,p,q) + \lambda F_R(w).$$

Since Π_K is compact, $\widehat{H}^K$ is finite and continuous. Hence we are in the regular compact-control case[1] of [Pham, 2009, Theorem 4.3.1 and Remark 4.3.4] and the variational inequality reduces to the ordinary HJB equation. The argument in [Pham, 2009, Section 4.3 and Theorem 4.3.1] therefore applies in the present setting and gives that $V^{R,K}$ is a viscosity solution of

$$\lambda V^{R,K}(w) - \widehat{H}^K\big(w, (V^{R,K})'(w), (V^{R,K})''(w)\big) = 0$$

on $\mathbb{R}$ and hence, also on $(0, b_R)$. Substituting the expression for $\widehat{H}^K$ gives

$$\lambda V^{R,K}(w) - H^K\big(w, (V^{R,K})'(w), (V^{R,K})''(w)\big) - \lambda F_R(w) = 0$$

in the viscosity sense on $(0, b_R)$ as required.

Step 3: (Uniqueness) Let u and v be two bounded continuous viscosity solutions of (25) on $(0, b_R)$ with boundary values $u(0) = v(0) = 0$ and $u(b_R) = v(b_R) = 1$. Applying Lemma S.1.5 first to u as subsolution and v as supersolution gives $u \leq v$. Reversing the roles gives $v \leq u$. Hence $u = v$ on $[0, b_R]$. □

[1] We note that the infinite-horizon viscosity result is stated under the condition that (in our notation) λ is "sufficiently large." That condition is used in [Pham, 2009, Proposition 4.3.1] to ensure that the infinite-horizon payoff is well-defined in the unbounded reward setting, and to ensure that constant controls are admissible (see the invocation of [Pham, 2009, Remark 3.2.2] in the proof). In the present problem these requirements hold for every $\lambda > 0$, because the running reward is uniformly bounded over the admissible controls $\boldsymbol{\pi}$ and so we can dispense with this requirement.

S.1.3 Dual-Goal Problem

Throughout this subsection we use Proposition 5.1 and work with the reduced ex ante formulation of the dual-goal value. Thus, for $(t,w) \in [0,T) \times [0,\infty)$,

$$V^{R,D,K}(t,w) = \widetilde{V}^{R,D,K}(t,w) = \sup_{\boldsymbol{\pi}\in\mathcal{A}^{K,\mathrm{pre}}_{t,T}} \mathbb{E}_{t,w}\Bigg[\int_t^T \lambda e^{-\lambda(s-t)} \mathcal{J}^K\Big(s, \widetilde{W}^{t,w;\boldsymbol{\pi}}_s\Big)\, ds + e^{-\lambda(T-t)}\mathcal{T}^K\Big(\widetilde{W}^{t,w;\boldsymbol{\pi}}_T\Big)\Bigg], \tag{S.2}$$

where $\widetilde{W}^{t,w;\boldsymbol{\pi}}$ is the pre-payment wealth process

$$d\widetilde{W}^{t,w;\boldsymbol{\pi}}_s = \widetilde{W}^{t,w;\boldsymbol{\pi}}_s \Big(\boldsymbol{\pi}_s^\top \boldsymbol{\theta}\, ds + \boldsymbol{\pi}_s^\top \boldsymbol{\sigma}\, dB_s\Big), \qquad \widetilde{W}^{t,w;\boldsymbol{\pi}}_t = w.$$

This is a standard finite-horizon stochastic control problem with discount rate λ, running reward $\lambda\mathcal{J}^K$, and terminal reward $\mathcal{T}^K$.

Recall that $b := b_R + b_G$. We first record the regularity of the two payoff operators. For $w \geq 0$, these operators are

$$\mathcal{J}^K(t,w) = \mathbb{E}_R\big[\alpha_R \mathbf{1}_{\{w\geq R\}} + \alpha_D V^{D,K}\big(t, w - R\mathbf{1}_{\{w\geq R\}}\big)\big]\,,$$

and

$$\mathcal{T}^K(w) = \mathbb{E}_G\big[\alpha_D \mathbf{1}_{\{w\geq G\}} + \alpha_R V^{R,K}\big(w - G\mathbf{1}_{\{w\geq G\}}\big)\big]\,.$$

For the proof below, we extend them harmlessly to all real w by setting

$$\mathcal{J}^K(t,w) = \mathcal{T}^K(w) = 0 \quad \text{for } w \leq 0.$$

For $w \geq b$, the formulas above already give

$$\mathcal{J}^K(t,w) = \mathcal{T}^K(w) = 1,$$

because $R \leq b_R$, $G \leq b_G$, $V^{D,K}(t,x) = 1$ for $x \geq b_G$, and $V^{R,K}(x) = 1$ for $x \geq b_R$.

Lemma S.1.6. *The functions $\mathcal{J}^K$ and $\mathcal{T}^K$ are bounded and continuous: $\mathcal{J}^K \in C([0,T]\times\mathbb{R})$, $\mathcal{T}^K \in C(\mathbb{R})$. Moreover, both functions take values in $[0,1]$, and both are constant outside the compact wealth interval $[0,b]$.*

Proof. The boundedness is immediate from $0 \leq V^{D,K}, V^{R,K} \leq 1$ and $\alpha_R + \alpha_D = 1$.

We prove continuity of $\mathcal{J}^K$; the proof for $\mathcal{T}^K$ is the same. Let $(t_n, w_n) \to (t,w)$ in $[0,T]\times\mathbb{R}$. It is enough to consider $w \in [0,b]$, since $\mathcal{J}^K$ is constant for $w \leq 0$ and for $w \geq b$. By Proposition 4.1, $V^{D,K}$ is continuous on $[0,T]\times[0,\infty)$, and it is extended constantly as 1 for wealth at least b_G.

For every realization r such that $r \neq w$, we have

$$\mathbf{1}_{\{w_n\geq r\}} \to \mathbf{1}_{\{w\geq r\}}, \qquad w_n - r\mathbf{1}_{\{w_n\geq r\}} \to w - r\mathbf{1}_{\{w\geq r\}}.$$

Because F_R is continuous, $\mathbb{P}(R = w) = 0$. Hence the integrand in the definition of $\mathcal{J}^K(t_n, w_n)$ converges almost surely to the integrand in the definition of $\mathcal{J}^K(t, w)$. The integrands are bounded by 1, so dominated convergence gives

$$\mathcal{J}^K(t_n, w_n) \to \mathcal{J}^K(t, w).$$

This proves $\mathcal{J}^K \in C([0, T] \times \mathbb{R})$. The proof for $\mathcal{T}^K$ uses Proposition 4.2 and the continuity of F_G, and is identical. □

Lemma S.1.7. *The value function $\widetilde{V}^{R,D,K}$ defined by (30) is continuous on $[0, T] \times [0, \infty)$.*

Proof. By Lemma S.1.6, $\mathcal{J}^K$ and $\mathcal{T}^K$ are bounded and uniformly continuous after the harmless constant extensions described above. Therefore there exist bounded Lipschitz functions

$$\mathcal{J}_m : [0, T] \times \mathbb{R} \to [0, 1], \qquad \mathcal{T}_m : \mathbb{R} \to [0, 1],$$

such that

$$\|\mathcal{J}_m - \mathcal{J}^K\|_\infty + \|\mathcal{T}_m - \mathcal{T}^K\|_\infty \longrightarrow 0.$$

For each m, define the auxiliary value function

$$V_m(t, w) := \sup_{\boldsymbol{\pi} \in \mathcal{A}^{K,\mathrm{pre}}_{t,T}} \mathbb{E}_{t,w} \left[\int_t^T \lambda e^{-\lambda(s-t)} \mathcal{J}_m\left(s, \widetilde{W}^{t,w;\boldsymbol{\pi}}_s\right) \, ds + e^{-\lambda(T-t)} \mathcal{T}_m\left(\widetilde{W}^{t,w;\boldsymbol{\pi}}_T\right) \right].$$

The coefficients

$$b(w, \boldsymbol{\pi}) = w \boldsymbol{\pi}^\top \boldsymbol{\theta}, \qquad \sigma_W(w, \boldsymbol{\pi}) = w \boldsymbol{\pi}^\top \boldsymbol{\sigma}$$

are globally Lipschitz in w, uniformly over $\boldsymbol{\pi} \in \Pi_K$, and satisfy a uniform linear-growth bound.

We now justify the use of [Touzi, 2013, Proposition 3.7]. That proposition is stated after reducing, for simplicity, to the Mayer case $f = k \equiv 0$. Since the horizon is finite, we may use the standard augmentation described in [Touzi, 2013, Remark 3.2(iii)]. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function such that $\chi(z) = z$ for $z \in [e^{-\lambda T}, 1]$. For the auxiliary problem define additional state variables Y and Z by

$$dY_s = \chi(Z_s)\, \lambda \mathcal{J}_m\left(s, \widetilde{W}^{t,w;\boldsymbol{\pi}}_s\right) \, ds, \qquad Y_t = 0,$$

and

$$dZ_s = -\lambda Z_s \, ds, \qquad Z_t = 1.$$

Then $Z_s = e^{-\lambda(s-t)} \in [e^{-\lambda T}, 1]$, so $\chi(Z_s) = Z_s$ on every reachable path starting from $Z_t = 1$. Hence the payoff in the definition of V_m can be written in Mayer form as

$$Y_T + \chi(Z_T) \mathcal{T}_m\left(\widetilde{W}^{t,w;\boldsymbol{\pi}}_T\right).$$

The augmented coefficients are Lipschitz in the state and have linear growth, uniformly over the bounded control set Π_K. Indeed, χ, $\mathcal{J}_m$, and $\mathcal{T}_m$ are bounded and Lipschitz. The augmented terminal payoff

$$(w, y, z) \longmapsto y + \chi(z)\mathcal{T}_m(w)$$

is also Lipschitz. Therefore [Touzi, 2013, Proposition 3.7] applies to the augmented Mayer problem, and each V_m is continuous on $[0, T] \times \mathbb{R}$.

For every (t, w),

$$\begin{aligned}\left|V_m(t,w) - \widetilde{V}^{R,D,K}(t,w)\right| &\le \sup_{\boldsymbol{\pi}} \mathbb{E}_{t,w}\left[\int_t^T \lambda e^{-\lambda(s-t)} \left|\mathcal{J}_m - \mathcal{J}^K\right|\left(s, \widetilde{W}_s^{t,w;\boldsymbol{\pi}}\right) ds\right] \\ &\quad + \sup_{\boldsymbol{\pi}} \mathbb{E}_{t,w}\left[e^{-\lambda(T-t)} \left|\mathcal{T}_m - \mathcal{T}^K\right|\left(\widetilde{W}_T^{t,w;\boldsymbol{\pi}}\right)\right] \\ &\le \|\mathcal{J}_m - \mathcal{J}^K\|_\infty + \|\mathcal{T}_m - \mathcal{T}^K\|_\infty.\end{aligned}$$

Hence $V_m \to \widetilde{V}^{R,D,K}$ uniformly on $[0,T] \times [0,\infty)$. Since $\widetilde{V}^{R,D,K}$ is the uniform limit of continuous functions, it is continuous. □

Define the dual-goal operator

$$\mathcal{F}_{RD}^K(t, w, r, a, p, q) := \lambda r - a - H^K(w, p, q) - \lambda \mathcal{J}^K(t, w).$$

Thus $\mathcal{F}_{RD}^K = 0$ is equivalent to

$$u_t(t,w) + H^K\big(w, u_w(t,w), u_{ww}(t,w)\big) + \lambda\big(\mathcal{J}^K(t,w) - u(t,w)\big) = 0.$$

With the sign convention used in the single-goal proofs, a viscosity subsolution of $\mathcal{F}_{RD}^K = 0$ satisfies $\mathcal{F}_{RD}^K \le 0$, and a viscosity supersolution satisfies $\mathcal{F}_{RD}^K \ge 0$.

Lemma S.1.8 (Comparison)**.** *Let $Q := (0,T) \times (0,b)$. Let $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$ be bounded. Suppose that u is a viscosity subsolution and v is a viscosity supersolution of*

$$\mathcal{F}_{RD}^K(t, w, z, z_t, z_w, z_{ww}) = 0 \qquad \text{in } Q.$$

Assume that the parabolic boundary inequalities

$$u(T, w) \le \mathcal{T}^K(w) \le v(T, w), \qquad 0 \le w \le b,$$

and

$$u(t, 0) \le 0 \le v(t, 0), \qquad u(t, b) \le 1 \le v(t, b), \qquad 0 \le t \le T,$$

hold. Then

$$u(t, w) \le v(t, w) \qquad \text{for all } (t, w) \in (0, T] \times [0, b].$$

If, in addition, u and v are continuous on $\overline{Q}$, then the inequality holds on all of $[0, T] \times [0, b]$.

Proof. We reduce the result to the parabolic Cauchy–Dirichlet comparison theorem [Crandall et al., 1992, Theorem 8.2], exactly as in Lemma S.1.3. Set $s := T - t,\ h(w) := \frac{w}{b}$, and define

$$U(s,w) := u(T-s,w) - h(w), \qquad V(s,w) := v(T-s,w) - h(w).$$

The boundary inequalities become

$$U(0,w) \le \mathcal{T}^K(w) - h(w) \le V(0,w), \qquad 0 \le w \le b,$$

and

$$U(s,0) \le 0 \le V(s,0), \qquad U(s,b) \le 0 \le V(s,b), \qquad 0 \le s \le T.$$

We next identify the transformed equation. If $\psi \in C^{1,2}$ is a test function for U, the corresponding test function for u is

$$\phi(t,w) := \psi(T-t,w) + h(w).$$

Hence

$$\phi_t(t,w) = -\psi_s(T-t,w), \qquad \phi_w(t,w) = \psi_w(T-t,w)+h'(w), \qquad \phi_{ww}(t,w) = \psi_{ww}(T-t,w)+h''(w).$$

Since $h'(w) = 1/b$ and $h''(w) = 0$, the viscosity inequalities for u and v transform into the viscosity inequalities for the Cauchy–Dirichlet problem

$$z_s + \widehat{F}(s,w,z,z_w,z_{ww}) = 0 \qquad \text{in } (0,T)\times(0,b),$$

where

$$\widehat{F}(s,w,r,p,q) := \lambda\big(r + h(w)\big) - H^K\left(w, p + \frac{1}{b}, q\right) - \lambda \mathcal{J}^K(T-s,w).$$

It remains only to verify the structural hypotheses of [Crandall et al., 1992, Theorem 8.2]. By Lemma S.1.1 and Lemma S.1.6, $\widehat{F}$ is continuous. It is proper because it is nondecreasing in the value variable r, with slope $\lambda > 0$, and it is degenerate elliptic because H^K is nondecreasing in its second-derivative argument q. Indeed, if $q_1 \le q_2$, then

$$H^K(w,p,q_1) \le H^K(w,p,q_2),$$

and therefore

$$\widehat{F}(s,w,r,p,q_1) \ge \widehat{F}(s,w,r,p,q_2).$$

We now verify the structural condition [Crandall et al., 1992, Equation (3.14)]. For $\boldsymbol{\pi} \in \Pi_K$, set

$$\beta^{\boldsymbol{\pi}}(w) := -w\,\boldsymbol{\pi}^\top\boldsymbol{\theta}, \qquad Z^{\boldsymbol{\pi}}(w) := \frac{w}{\sqrt{2}}\big(\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi}\big)^{1/2}, \qquad f^{\boldsymbol{\pi}}(w) := \frac{w}{b}\,\boldsymbol{\pi}^\top\boldsymbol{\theta}.$$

Then, identifying $Z^{\boldsymbol{\pi}}(w)$ with a 1×1 matrix,

$$\widehat{F}(s,w,r,p,q) = \lambda r + \lambda h(w) - \lambda \mathcal{J}^K(T-s,w) + \inf_{\boldsymbol{\pi}\in\Pi_K} \left\{ \beta^{\boldsymbol{\pi}}(w)p - \operatorname{tr}\Big(Z^{\boldsymbol{\pi}}(w)^\top Z^{\boldsymbol{\pi}}(w)q\Big) - f^{\boldsymbol{\pi}}(w) \right\}.$$

For each fixed $\boldsymbol{\pi}$, the first-order term, the second-order diffusion term, and the continuous spatial terms are covered by [Crandall et al., 1992, Example 3.6]. Because Π_K is compact, the one-sided Lipschitz constants for $\beta^{\boldsymbol{\pi}}$, the Lipschitz constants for $Z^{\boldsymbol{\pi}}$, and the moduli of continuity of $f^{\boldsymbol{\pi}}$ can be chosen uniformly over $\boldsymbol{\pi} \in \Pi_K$. Also, $\mathcal{J}^K$ is uniformly continuous on the compact set $[0,T] \times [0,b]$, so the modulus of continuity for the spatial term $\lambda h(w) - \lambda \mathcal{J}^K(T-s,w)$ can be chosen uniformly in s. The stability of the structural condition under sums and infima, stated in [Crandall et al., 1992, Example 3.6], therefore implies that $\widehat{F}$ satisfies the same condition, uniformly in $s \in [0,T]$.

The comparison theorem [Crandall et al., 1992, Theorem 8.2] therefore gives

$$U(s,w) \le V(s,w) \qquad \text{for } (s,w) \in [0,T) \times [0,b].$$

Returning to $t = T - s$, we obtain

$$u(t,w) \le v(t,w) \qquad \text{for } (t,w) \in (0,T] \times [0,b].$$

If u and v are continuous on $\overline{Q}$, the inequality extends to $t = 0$ by continuity. $\square$

We are now ready to complete the proof of our main theorem.

Proof of Theorem 5.2. We proceed in stages.

Step 1: (Regularity and Boundary Values) Since Lemma S.1.7 tells us that $\widetilde{V}^{R,D,K}$ is continuous on $[0,T] \times [0,\infty)$ and by Proposition 5.1 $\widetilde{V}^{R,D,K} = V^{R,D,K}$ on $[0,T) \times [0,\infty)$, the value function $V^{R,D,K}$ is continuous on $[0,T) \times [0,\infty)$. Moreover, the continuous extension of $V^{R,D,K}$ to $[0,T] \times [0,\infty)$ agrees with $\widetilde{V}^{R,D,K}$. It is bounded between 0 and 1, because the objective is a weighted sum of two success indicators and $\alpha_R + \alpha_D = 1$.

The boundary values are immediate from the reduced representation. If $w = 0$, then the multiplicative wealth process remains identically zero, and since $F_R(0) = F_G(0) = 0$, $V^{R,D,K}(t,0) = 0$. If $w \ge b = b_R + b_G$, the household may choose $\boldsymbol{\pi} \equiv 0$. Then it can fund whichever goal arrives first and still has enough wealth to fund the remaining goal. Since the value cannot exceed 1, this gives

$$V^{R,D,K}(t,w) = 1, \qquad 0 \le t \le T, \quad w \ge b.$$

Finally, if $t = T$, the running interval in (S.2) is empty and $\widetilde{W}_T^{T,w;\boldsymbol{\pi}} = w$, so the continuous extension $\widetilde{V}^{R,D,K}$ satisfies

$$\widetilde{V}^{R,D,K}(T,w) = \mathcal{T}^K(w), \qquad 0 \le w \le b.$$

Step 2: (Viscosity Solution) We now identify the (continuous extension of the) value function with a solution to the viscosity equation. The reduced control problem (S.2) is a finite-horizon stochastic control problem with state coefficients

$$b(w,\boldsymbol{\pi}) = w\boldsymbol{\pi}^\top\boldsymbol{\theta}, \qquad \sigma_W(w,\boldsymbol{\pi}) = w\boldsymbol{\pi}^\top\boldsymbol{\sigma},$$

discount rate $k \equiv \lambda$, running reward $f(t,w,\boldsymbol{\pi}) = \lambda\mathcal{J}^K(t,w)$, and terminal reward $g(w) = \mathcal{T}^K(w)$. The coefficients are Lipschitz in the state and have linear growth, uniformly over the compact control set Π_K. The functions f and g are bounded and continuous by Lemma S.1.6, and $k \equiv \lambda$ is bounded. The associated Hamiltonian from [Touzi, 2013, Section 7] is

$$\mathfrak{H}(t,w,r,p,q) := \sup_{\boldsymbol{\pi}\in\Pi_K} \left\{ -\lambda r + wp\,\boldsymbol{\pi}^\top\boldsymbol{\theta} + \frac{1}{2}w^2 q\,\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi} + \lambda\mathcal{J}^K(t,w) \right\}.$$

Because Π_K is compact and $\mathcal{J}^K$ is continuous, $\mathfrak{H}$ is finite and continuous. Since $\widetilde{V}^{R,D,K}$ is bounded, it is locally bounded. Hence the hypotheses of the finite-horizon viscosity characterization theorem [Touzi, 2013, Theorem 7.4] are satisfied, and $\widetilde{V}^{R,D,K}$ is a viscosity solution of

$$\lambda u(t,w) - u_t(t,w) - \sup_{\boldsymbol{\pi}\in\Pi_K} \left\{ wu_w(t,w)\boldsymbol{\pi}^\top\boldsymbol{\theta} + \frac{1}{2}w^2 u_{ww}(t,w)\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi} + \lambda\mathcal{J}^K(t,w) \right\} = 0.$$

Equivalently,

$$u_t(t,w) + H^K\big(w, u_w(t,w), u_{ww}(t,w)\big) + \lambda\big(\mathcal{J}^K(t,w) - u(t,w)\big) = 0$$

in $\Omega_{RD} = (0,T)\times(0,b)$, in the viscosity sense.

<u>*Step 3:*</u> *(Uniqueness)* It remains to prove uniqueness. Let U and W be two bounded continuous viscosity solutions of

$$\partial_t z + H^K(w, z_w, z_{ww}) + \lambda(\mathcal{J}^K - z) = 0 \qquad \text{in } \Omega_{RD},$$

with boundary data

$$z(T,w) = \mathcal{T}^K(w), \quad 0 \le w \le b, \qquad z(t,0) = 0, \quad z(t,b) = 1, \quad 0 \le t \le T.$$

Applying Lemma S.1.8 to U as subsolution and W as supersolution gives $U \le W$ on $[0,T]\times[0,b]$. Reversing the roles gives $W \le U$. Hence $U = W$.

Thus the continuous extension of $V^{R,D,K}$, restricted to $[0,T]\times[0,b]$, is the unique bounded continuous viscosity solution of (31)–(32). □

S.2 Proof of Proposition 5.1

S.2.1 Conventions and Preliminaries

To make the conditioning arguments below formal, we take the model on a canonical product realization, for example $C([0,\infty);\mathbb{R}^d)\times[0,\infty)\times[0,b_R]\times[0,b_G]$, carrying the Brownian coordinate, the exponential clock, and the two goal amounts, and then pass to the usual completed, right-continuous filtrations. Regular conditional probabilities below are taken on the raw countably generated sigma-fields and then identified with their completed versions. Predictable controls are identified up to $ds \otimes d\mathbb{P}$-null sets with raw predictable versions. Naturally, conclusions about the value function remain wholly unchanged by this choice of convention.

We will always work *conditionally on* $\{\tau > t\}$ and write $\mathbb{E}_{t,w} := \mathbb{E}[\,\cdot \mid \tau > t\,]$ for the conditional expectation with pre-payment wealth started at w at time t. On $\{\tau > t\}$ set $\tau_t := \tau$. By the memorylessness of the exponential distribution and the independence of τ from $\mathcal{F}_\infty \vee \sigma(R, G)$, under $\mathbb{P}(\cdot \mid \tau > t)$ the residual time $\tau_t - t$ is $\mathrm{Exp}(\lambda)$ and is independent of $\mathcal{F}_\infty \vee \sigma(R, G)$; in particular $\mathbb{P}(\tau_t = T \mid \tau > t) = 0$ and $\mathbb{P}(\tau_t > T \mid \tau > t) = e^{-\lambda(T-t)}$. For convenience we adopt the same notation for all-or-nothing post-payment map as in the motivating section:

$$\Gamma_a(x) := x - a\,\mathbf{1}_{\{x \geq a\}}, \qquad a \geq 0,\ x \geq 0. \tag{S.3}$$

Recall that for $\boldsymbol{\pi} \in \mathcal{A}^{K,\mathrm{pre}}_{t,u}$ (or $\boldsymbol{\pi} \in \mathcal{A}^{K}_{t,u}$) the pre-payment wealth $\widetilde{W}^{t,w;\boldsymbol{\pi}}$ is the strong solution of

$$d\widetilde{W}^{t,w;\boldsymbol{\pi}}_s = \widetilde{W}^{t,w;\boldsymbol{\pi}}_s \big(\boldsymbol{\pi}_s^\top \boldsymbol{\theta}\, ds + \boldsymbol{\pi}_s^\top \boldsymbol{\sigma}\, dB_s\big), \qquad \widetilde{W}^{t,w;\boldsymbol{\pi}}_t = w, \quad s \in [t, u]. \tag{S.4}$$

The post-payment wealth $\widehat{W}^{t,w;\boldsymbol{\pi}}$ for a full strategy $\boldsymbol{\pi} \in \mathcal{A}^K_{t,\infty}$ is the unique càdlàg $\mathbb{G}$-adapted solution of

$$\widehat{W}^{t,w;\boldsymbol{\pi}}_s = w + \int_t^s \widehat{W}^{t,w;\boldsymbol{\pi}}_{r-}\, \boldsymbol{\pi}_r^\top \boldsymbol{\theta}\, dr + \int_t^s \widehat{W}^{t,w;\boldsymbol{\pi}}_{r-}\, \boldsymbol{\pi}_r^\top \boldsymbol{\sigma}\, dB_r - R\,\mathbf{1}_{\{\tau \leq s,\ \widehat{W}^{t,w;\boldsymbol{\pi}}_{\tau-} \geq R\}} - G\,\mathbf{1}_{\{T \leq s,\ \widehat{W}^{t,w;\boldsymbol{\pi}}_{T-} \geq G\}}, \tag{S.5}$$

$s \geq t$. Thus $\widehat{W}^{t,w;\boldsymbol{\pi}}$ follows (S.4) between goal times and has all-or-nothing jumps at τ and T.

We remark that for $s \in [0, T]$ and $x \geq 0$, we may write

$$\begin{aligned}
\mathcal{J}^K(s, x) &:= \mathbb{E}_R\big[\alpha_R \mathbf{1}_{\{x \geq R\}} + \alpha_D V^{D,K}(s, \Gamma_R(x))\big] \qquad \text{(S.6)}\\
&= \alpha_R F_R(x) + \alpha_D \left[(1 - F_R(x)) V^{D,K}(s, x) + \int_0^x V^{D,K}(s, x - r)\, dF_R(r)\right],\\
\mathcal{T}^K(x) &:= \mathbb{E}_G\big[\alpha_D \mathbf{1}_{\{x \geq G\}} + \alpha_R V^{R,K}(\Gamma_G(x))\big] \qquad \text{(S.7)}\\
&= \alpha_D F_G(x) + \alpha_R \left[(1 - F_G(x)) V^{R,K}(x) + \int_0^x V^{R,K}(x - g)\, dF_G(g)\right],
\end{aligned}$$

where $\mathbb{E}_R, \mathbb{E}_G$ average over the laws of R and G only.

We use throughout the following properties, established in the single-goal analysis: $V^{D,K}$ and $V^{R,K}$ take values in $[0, 1]$; $V^{D,K}$ is continuous on $[0, T] \times [0, \infty)$ and $V^{R,K}$ is continuous on $[0, \infty)$; and $V^{D,K}(s, x) = 1$ for $x \geq b_G$, $V^{R,K}(x) = 1$ for $x \geq b_R$. With $\sigma_\lambda \sim \mathrm{Exp}(\lambda)$ independent of $\mathcal{F}_\infty$, the elementary identity $\mathbb{E}[\int_0^\infty \lambda e^{-\lambda u} h(u)\, du] = \mathbb{E}[h(\sigma_\lambda)]$ applied to $h(u) = F_R(\widetilde{W}^{0,x;\boldsymbol{\pi}}_u)$ gives the equivalent form

$$V^{R,K}(x) = \sup_{\boldsymbol{\pi} \in \mathcal{A}^{K,\mathrm{pre}}_{0,\infty}} \mathbb{E}\left[F_R\left(\widetilde{W}^{0,x;\boldsymbol{\pi}}_{\sigma_\lambda}\right)\right]. \tag{S.8}$$

As noted in Lemma S.1.6 $(s,x) \mapsto \mathcal{J}^K(s,x)$ is continuous on $[0,T]\times[0,\infty)$ and $x \mapsto \mathcal{T}^K(x)$ is continuous on $[0,\infty)$. Hence for $\boldsymbol{\pi} \in \mathcal{A}_{t,T}^{K,\text{pre}}$ the process $(s,\omega) \mapsto \mathcal{J}^K(s, \widetilde{W}_s^{t,w;\boldsymbol{\pi}})$ is progressively measurable and bounded by 1, $\mathcal{T}^K(\widetilde{W}_T^{t,w;\boldsymbol{\pi}})$ is bounded by 1, and all expressions are finite.

Finally, the dual-goal value functions (and the focus of this appendix), are recorded again here for ease of reference. Namely, for $0 \le t \le T$ and $w \ge 0$,

$$V^{R,D,K}(t,w) := \sup_{\boldsymbol{\pi}\in\mathcal{A}_{t,\infty}^K} \mathbb{E}_{t,w}\left[\alpha_R \mathbf{1}_{\{\widehat{W}_{\tau_t-}^{t,w;\boldsymbol{\pi}} \ge R\}} + \alpha_D \mathbf{1}_{\{\widehat{W}_{T-}^{t,w;\boldsymbol{\pi}} \ge G\}}\right], \tag{S.9}$$

$$\widetilde{V}^{R,D,K}(t,w) := \sup_{\boldsymbol{\pi}\in\mathcal{A}_{t,T}^{K,\text{pre}}} \mathbb{E}_{t,w}\left[\int_t^T \lambda e^{-\lambda(s-t)} \mathcal{J}^K\left(s, \widetilde{W}_s^{t,w;\boldsymbol{\pi}}\right) ds + e^{-\lambda(T-t)} \mathcal{T}^K\left(\widetilde{W}_T^{t,w;\boldsymbol{\pi}}\right)\right]. \tag{S.10}$$

S.2.1.1 Additional Notation

- For a sigma-field $\mathcal{H}$ on Ω and a set $A \subseteq \Omega$, we write $\mathcal{H}|_A := \{H \cap A : H \in \mathcal{H}\}$ for the trace of $\mathcal{H}$ on A. We say that two sigma-fields $\mathcal{H}$ and $\mathcal{K}$ coincide on A if $\mathcal{H}|_A = \mathcal{K}|_A$. Equalities of completed sigma-fields below are understood up to completion.

- For stopping times S_1, S_2, we use the standard stochastic-interval notation $[\![S_1, S_2]\!] := \{(\omega, u) : S_1(\omega) \le u \le S_2(\omega)\}$, with the analogous meanings for $]\!]S_1, S_2]\!]$, $[\![S_1, S_2[\![$, and $]\!]S_1, S_2[\![$.

- Let $\mathbb{H} = (\mathcal{H}_t)_{t\ge0}$ be a filtration satisfying the usual conditions with respect to which B is a d-dimensional Brownian motion. If S is an a.s. finite $\mathbb{H}$-stopping time, we write $B_v^S := B_{S+v} - B_S$, $v \ge 0$, for the Brownian increment process after S. Its augmented natural filtration is denoted by $\mathbb{B}^S = (\mathcal{B}_v^S)_{v\ge0}$. By the strong Markov property, B^S is a d-dimensional Brownian motion independent of $\mathcal{H}_S$.

S.2.2 Auxiliary Results

A number of auxilliary results are needed for the main proof. We provide them here and motivate each in turn.

S.2.2.1 Predictable Reductions

The first result is a general result that will be used to justify a reduction in the effective class of controls from $\mathcal{A}_{t,\infty}^K$ to $\mathcal{A}_{t,\infty}^{K,\text{pre}}$.

Lemma S.2.1. *Let $\mathbb{H} = (\mathcal{H}_s)_{s\ge0}$ satisfy the usual conditions, let ρ be a random time, and let M be a real-valued random variable. Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{H}$ which makes ρ a stopping time and reveals M at ρ, i.e.*

$$\mathcal{G}_s = \bigcap_{u>s} \left(\mathcal{H}_u \vee \sigma(\rho \wedge u) \vee \sigma(M\mathbf{1}_{\{\rho\le u\}})\right).$$

Then every (possibly vector-valued) $\mathbb{G}$-predictable process Y admits an $\mathbb{H}$-predictable process Y' such that $Y = Y'$ on $]\!]0, \rho]\!]$ up to evanescence. Consequently,

$$Y = Y' \quad ds \otimes d\mathbb{P}\text{-a.e. on } [\![0, \rho]\!].$$

Proof. For the progressive enlargement without M, this is the result of [Aksamit and Jeanblanc, 2017, Proposition 2.11(b)]. The case with M follows from the same monotone-class argument. Indeed, M contributes no predictable information on $]\!]0, \rho]\!]$ so every $\mathbb{G}$-predictable process has an $\mathbb{H}$-predictable reduction on $]\!]0, \rho]\!]$, up to evanescence. The stated $ds \otimes d\mathbb{P}$ conclusion follows because the only difference between $]\!]0, \rho]\!]$ and $[\![0, \rho]\!]$ is the time-zero slice, which is $ds \otimes d\mathbb{P}$-null. □

As noted above, the next two lemmas use this general result in the context of our problem to argue that we may restrict to pre-arrival controls before goals have arrived.

Lemma S.2.2. *Fix $0 \le t < T$ and work on $\{\tau > t\}$. For every $\boldsymbol{\pi} \in \mathcal{A}^K_{t,\infty}$ there exists $\boldsymbol{\pi}^0 \in \mathcal{A}^{K,\mathrm{pre}}_{t,T}$ such that*

$$\boldsymbol{\pi}_s = \boldsymbol{\pi}^0_s \quad \text{for } ds \otimes d\mathbb{P}\text{-a.e. } (s, \omega) \text{ with } t \le s < \tau \wedge T.$$

Consequently, if $\widehat{W} = \widehat{W}^{t,w;\boldsymbol{\pi}}$ and $\widetilde{W} = \widetilde{W}^{t,w;\boldsymbol{\pi}^0}$, then almost surely $\widehat{W}_s = \widetilde{W}_s$ for $s \in [t, \tau \wedge T)$, and

$$\widehat{W}_{\tau-} = \widetilde{W}_\tau \quad \text{on } \{\tau < T\}, \qquad \widehat{W}_{T-} = \widetilde{W}_T \quad \text{on } \{\tau > T\}.$$

Proof. On $[0, T)$ the component $\sigma(G\mathbf{1}_{\{T \le u\}})$ in the definition of $\mathbb{G}$ is trivial. Thus, before T, the filtration $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ by τ and the process $R\mathbf{1}_{\tau \le u}$. Applying Lemma S.2.1 to $\boldsymbol{\pi}\mathbf{1}_{[t,T)}$ gives an $\mathbb{F}$-predictable process $\boldsymbol{\rho}$ which coincides with $\boldsymbol{\pi}$ on $\{(s, \omega) : 0 \le s < \tau(\omega)\} \cap ([t, T) \times \Omega)$ up to evanescence. Let $\boldsymbol{\pi}^0 := \mathrm{proj}_{\Pi_K} \boldsymbol{\rho}$ on $[t, T]$. Since the Euclidean projection onto the nonempty closed convex set Π_K is Borel and 1-Lipschitz, $\boldsymbol{\pi}^0$ is $\mathbb{F}$-predictable, Π_K-valued, and still coincides with $\boldsymbol{\pi}$ where $\boldsymbol{\pi}$ already takes values in Π_K. This gives the asserted equality of controls.

Before the first goal resolution no jump in (S.5) has occurred, so $\widehat{W}$ solves the diffusion equation (S.4) driven by $\boldsymbol{\pi}$ on $[t, \tau \wedge T)$. The stochastic and drift integrals up to any $s < \tau \wedge T$ depend on the control only through its restriction to $\{t \le r < s\}$, where $\boldsymbol{\pi} = \boldsymbol{\pi}^0$ $dr \otimes d\mathbb{P}$-a.e. Pathwise uniqueness for (S.4) gives $\widehat{W}_s = \widetilde{W}_s$ for $s < \tau \wedge T$, and the left-limit identities follow from continuity of $\widetilde{W}$. □

Lemma S.2.3. *Let $\mathbb{F}^G = (\mathcal{F}_s \vee \sigma(G))_{s \ge T}$ and augment it so that it is complete and right-continuous. For every $\boldsymbol{\pi} \in \mathcal{A}^K_{T,\infty}$ there exists an $\mathbb{F}^G$-predictable, Π_K-valued process $\boldsymbol{\pi}^G$ on $[T, \infty)$ such that*

$$\boldsymbol{\pi}_s = \boldsymbol{\pi}^G_s \quad \text{for } ds \otimes d\mathbb{P}\text{-a.e. } (s, \omega) \text{ with } T \le s < \tau.$$

Moreover, in relative time $v \ge 0$, the process $\boldsymbol{\pi}^G_{T+v}$ is predictable with respect to $(\mathcal{K}_T \vee \mathcal{B}^T_v)_{v \ge 0}$, where $\mathcal{K}_T := \mathcal{F}_T \vee \sigma(G)$ and $\mathcal{B}^T$ is the augmented natural filtration of $B^T = (B_{T+v} - B_T)_{v \ge 0}$.

Proof. Let $\mathbb{H} = (\mathcal{H}_s)_{s \ge 0}$ be the initially enlarged Brownian filtration $\mathcal{H}_s := \mathcal{F}_s \vee \sigma(G)$, completed and made right-continuous. On the time region $[T, \infty)$, the filtration $\mathbb{G}$ is the progressive enlargement of $\mathbb{H}$ by τ

and the process $R\mathbf{1}_{\{\tau\le u\}}$. Indeed, G is already contained in $\mathbb{H}$, while the pair (τ, R) is revealed only when τ occurs. Applying Lemma S.2.1 to the $\mathbb{G}$-predictable process $\boldsymbol{\pi}\mathbf{1}_{[T,\infty)}$ gives an $\mathbb{H}$-predictable process $\boldsymbol{\rho}$ which coincides with $\boldsymbol{\pi}$ on $[\![0,\tau]\!] \cap ([T,\infty)\times\Omega)$ up to evanescence. Restricting to $[T,\infty)$ and projecting onto Π_K yields an $\mathbb{F}^G$-predictable, Π_K-valued process $\boldsymbol{\pi}^G$ satisfying the asserted equality for $ds\otimes d\mathbb{P}$-a.e. (s,ω) with $T\le s<\tau$. On $\{\tau\le T\}$ this set is empty, so no additional assertion is needed.

Finally, since $\mathcal{F}_{T+v}=\mathcal{F}_T\vee\mathcal{B}_v^T$ up to null sets for Brownian natural filtrations, the initially enlarged filtration satisfies $\mathbb{F}^G_{T+v}=\mathcal{K}_T\vee\mathcal{B}_v^T$ up to completion. Hence the relative-time process $\boldsymbol{\pi}^G_{T+v}$ is predictable with respect to $(\mathcal{K}_T\vee\mathcal{B}_v^T)_{v\ge 0}$. □

S.2.2.2 Repesentations of Expectations

In our analysis, we will need to simplify expectations in terms of the exponential density. To this end, we supply the following two lemmas.

Lemma S.2.4. *Fix $0\le t<T$ and work on $\{\tau>t\}$. Let $\boldsymbol{\pi}\in\mathcal{A}_{t,T}^{K,\mathrm{pre}}$ and $\widetilde{W}=\widetilde{W}^{t,w;\boldsymbol{\pi}}$. For every bounded Borel $f:[t,T]\times[0,\infty)\to\mathbb{R}$,*

$$\mathbb{E}_{t,w}\Big[f\Big(\tau_t,\widetilde{W}_{\tau_t}\Big)\,\mathbf{1}_{\{\tau_t<T\}}\Big]=\int_t^T\lambda e^{-\lambda(s-t)}\,\mathbb{E}_{t,w}\Big[f\Big(s,\widetilde{W}_s\Big)\Big]\,ds,$$

and for every bounded Borel $g:[0,\infty)\to\mathbb{R}$,

$$\mathbb{E}_{t,w}\Big[g\Big(\widetilde{W}_T\Big)\,\mathbf{1}_{\{\tau_t>T\}}\Big]=e^{-\lambda(T-t)}\,\mathbb{E}_{t,w}\Big[g\Big(\widetilde{W}_T\Big)\Big].$$

Proof. Under $\mathbb{P}(\cdot\mid\tau>t)$ the residual $\eta=\tau_t-t$ has density $\lambda e^{-\lambda u}$ and is independent of $\mathcal{F}_\infty$. Since $\boldsymbol{\pi}$ is $\mathbb{F}$-predictable, the path $(\widetilde{W}_s)_{s\in[t,T]}$ is $\mathcal{F}_T$-measurable and the map $(s,\omega)\mapsto f(s,\widetilde{W}_s(\omega))$ is jointly measurable and bounded. By conditioning on η, using the density $\lambda e^{-\lambda u}$, the independence of η from $\mathcal{F}_T$, and Fubiniâ€™s theorem

$$\mathbb{E}_{t,w}\Big[f(t+\eta,\widetilde{W}_{t+\eta})\mathbf{1}_{\{\eta<T-t\}}\Big]=\int_0^{T-t}\lambda e^{-\lambda u}\,\mathbb{E}_{t,w}[f(t+u,\widetilde{W}_{t+u})]\,du.$$

The change of variables $s=t+u$ gives the first identity. The second follows from $\mathbb{P}(\eta>T-t\mid\tau>t)=e^{-\lambda(T-t)}$ and the independence of η from $\widetilde{W}_T$. □

Lemma S.2.5. *Fix $0\le t<T$ and work on $\{\tau>t\}$. Set $\mathcal{K}_T:=\mathcal{F}_T\vee\sigma(G)$ and let $\sigma_\lambda\sim\mathrm{Exp}(\lambda)$ be independent of $\mathcal{F}_\infty\vee\sigma(R,G)$. Let $(\psi(s))_{s\ge T}$ be a bounded and jointly measurable process, with $\psi(s)$ being $\sigma(\mathcal{F}_s,G,R)$-measurable for each $s\ge T$. Then, almost surely,*

$$\mathbb{E}_{t,w}\big[\mathbf{1}_{\{\tau_t>T\}}\,\psi(\tau_t)\,\big|\,\mathcal{K}_T\big]=e^{-\lambda(T-t)}\,\mathbb{E}[\psi(T+\sigma_\lambda)\,|\,\mathcal{K}_T].$$

Proof. On $\{\tau>t\}$ write $\tau_t=t+\eta$ with $\eta\sim\mathrm{Exp}(\lambda)$ independent of $\mathcal{F}_\infty\vee\sigma(R,G)\supseteq\mathcal{K}_T$. Since $\{\tau>t\}$

is independent of $\mathcal{F}_\infty \vee \sigma(R, G)$, for each $u \geq T - t$,

$$\mathbb{E}_{t,w}[\psi(t+u) \mid \mathcal{K}_T] = \mathbb{E}[\psi(t+u) \mid \mathcal{K}_T].$$

Hence, by the conditional version of Fubini's theorem,

$$\mathbb{E}_{t,w}\left[\mathbf{1}_{\{t+\eta>T\}}\psi(t+\eta) \,\middle|\, \mathcal{K}_T\right] = \int_{T-t}^{\infty} \mathbb{E}[\psi(t+u) \mid \mathcal{K}_T]\,\lambda e^{-\lambda u}\,du.$$

Substituting $u = (T-t) + v$ gives $\lambda e^{-\lambda u} = \lambda e^{-\lambda(T-t)} e^{-\lambda v}$ and $t + u = T + v$. Therefore the right-hand side equals

$$e^{-\lambda(T-t)} \int_0^{\infty} \mathbb{E}[\psi(T+v) \mid \mathcal{K}_T]\,\lambda e^{-\lambda v}\,dv = e^{-\lambda(T-t)}\mathbb{E}[\psi(T+\sigma_\lambda) \mid \mathcal{K}_T],$$

where the last equality again uses conditional Fubini and the independence of σ_λ from $\mathcal{F}_\infty \vee \sigma(R, G)$. $\square$

S.2.2.3 Alternative Representation of the Single Goal Value Functions

Next, we will also need an alternative way to interpret the single goal value functions. Let β be the coordinate process on $C([0,\infty);\mathbb{R}^d)$, let $\mathbb{W}$ be Wiener measure, and let $\mathbb{B} = (\mathcal{B}_v)_{v\geq 0}$ be the usual augmentation of the canonical Brownian filtration. Let $\mathcal{U}^K$ be the set of Π_K-valued $\mathbb{B}$-predictable processes on this canonical space. Thus each $\boldsymbol{\nu} \in \mathcal{U}^K$ is a predictable functional of a Brownian increment path. If W is any d-dimensional Brownian motion on any filtered probability space, we write $\boldsymbol{\nu}(W)$ for the process obtained by evaluating this functional on the path of W, and let $X^{x;\boldsymbol{\nu}}(W)$ solve

$$dX_v = X_v\big(\boldsymbol{\nu}_v(W)^\top \boldsymbol{\theta}\,dv + \boldsymbol{\nu}_v(W)^\top \boldsymbol{\sigma}\,dW_v\big), \qquad X_0 = x.$$

Define the following payoffs, with expectation under Wiener measure,

$$P_D^{\boldsymbol{\nu}}(s,x) := \mathbb{E}_{\mathbb{W}}\left[F_G\big(X_{T-s}^{x;\boldsymbol{\nu}}(\beta)\big)\right], \qquad P_R^{\boldsymbol{\nu}}(x) := \mathbb{E}_{\mathbb{W}}\left[\int_0^\infty \lambda e^{-\lambda u} F_R(X_u^{x;\boldsymbol{\nu}}(\beta))\,du\right].$$

Our next result relates these to $V^{D,K}$ and $V^{R,K}$.

Lemma S.2.6. *For all $(s,x) \in [0,T] \times [0,\infty)$,*

$$V^{D,K}(s,x) = \sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_D^{\boldsymbol{\nu}}(s,x), \qquad V^{R,K}(x) = \sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_R^{\boldsymbol{\nu}}(x).$$

Proof. We prove the fixed-deadline identity. The random deadline identity follows analogously.

Step 1: We first prove the inequality $\sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_D^{\boldsymbol{\nu}}(s,x) \leq V^{D,K}(s,x)$. Consider the Brownian motion $B^s = (B_{s+v} - B_s)_{v\geq 0}$. For $\boldsymbol{\nu} \in \mathcal{U}^K$ the shifted control $\boldsymbol{\pi}_u := \boldsymbol{\nu}_{u-s}(B^s)$, $u \in [s,T]$, is $\mathbb{F}$-predictable and Π_K-valued, so $\boldsymbol{\pi} \in \mathcal{A}_{s,T}^{K,\mathrm{pre}}$. Moreover, $\widetilde{W}_T^{s,x;\boldsymbol{\pi}} = X_{T-s}^{x;\boldsymbol{\nu}}(B^s)$. Since B^s has the Wiener law,

$$P_D^{\boldsymbol{\nu}}(s,x) = \mathbb{E}\left[F_G\left(\widetilde{W}_T^{s,x;\boldsymbol{\pi}}\right)\right] \leq V^{D,K}(s,x).$$

Taking the supremum over $\boldsymbol{\nu} \in \mathcal{U}^K$ gives the desired inequality.

Step 2: Conversely, we can show $V^{D,K}(s,x) \le \sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_D^{\boldsymbol{\nu}}(s,x)$. Fix $\boldsymbol{\pi} \in \mathcal{A}_{s,T}^{K,\mathrm{pre}}$. Let $\mathbb{P}^\omega := \mathbb{P}(\cdot \mid \mathcal{F}_s)(\omega)$ be a regular conditional probability given $\mathcal{F}_s$, chosen on the underlying raw standard-Borel space (cf. [Kallenberg, 2021, Thm. 8.5]). Since B^s is independent of $\mathcal{F}_s$, for $\mathbb{P}$-a.e. ω, under $\mathbb{P}^\omega$ the process B^s is a Brownian motion and every set in $\mathcal{F}_s$ has $\mathbb{P}^\omega$-probability 0 or 1.

By the standing raw-filtration convention, we choose a raw predictable version of $\boldsymbol{\pi}$. On the raw Brownian filtration one has $\mathcal{F}_{s+v} = \mathcal{F}_s \vee \mathcal{B}_v^s$ before completion, so the relative-time process $v \mapsto \boldsymbol{\pi}_{s+v}$ is predictable with respect to $(\mathcal{F}_s \vee \mathcal{B}_v^s)_{v\ge 0}$. We use the standard predictable factorization fact that, under a conditional law for which $\mathcal{F}_s$ is trivial, any Π_K-valued $(\mathcal{F}_s \vee \mathcal{B}_v^s)_{v\ge 0}$-predictable process is equal, up to $dv \otimes d\mathbb{P}^\omega$-null sets, to a canonical predictable functional $\boldsymbol{\nu}^\omega(B^s)$ for some $\boldsymbol{\nu}^\omega \in \mathcal{U}^K$.

Therefore, for $\mathbb{P}$-a.e. ω, there exists $\boldsymbol{\nu}^\omega \in \mathcal{U}^K$ such that

$$\boldsymbol{\pi}_{s+v} = \boldsymbol{\nu}_v^\omega(B^s) \qquad dv \otimes d\mathbb{P}^\omega\text{-a.e. on } [0, T-s] \times \Omega.$$

Under $\mathbb{P}^\omega$, the two controls $\boldsymbol{\pi}_{s+\cdot}$ and $\boldsymbol{\nu}^\omega(B^s)$ agree $dv \otimes d\mathbb{P}^\omega$-a.e. and are bounded. Hence the corresponding drift integrals coincide, and the Itô integrals coincide by the Itô isometry applied to the difference of the two bounded integrands. Pathwise uniqueness for the wealth equation then gives that $\widetilde{W}_T^{s,x;\boldsymbol{\pi}} = X_{T-s}^{x;\boldsymbol{\nu}^\omega}(B^s)$, $\mathbb{P}^\omega$-a.s. Consequently,

$$\mathbb{E}^{\mathbb{P}^\omega}\left[F_G\left(\widetilde{W}_T^{s,x;\boldsymbol{\pi}}\right)\right] = P_D^{\boldsymbol{\nu}^\omega}(s,x) \le \sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_D^{\boldsymbol{\nu}}(s,x).$$

Taking $\mathbb{P}$-expectations and using $\mathbb{E}[\,\cdot\,] = \mathbb{E}[\mathbb{E}^{\mathbb{P}^\cdot}[\,\cdot\,]]$ yields $\mathbb{E}\left[F_G\left(\widetilde{W}_T^{s,x;\boldsymbol{\pi}}\right)\right] \le \sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_D^{\boldsymbol{\nu}}(s,x)$. Taking the supremum over $\boldsymbol{\pi} \in \mathcal{A}_{s,T}^{K,\mathrm{pre}}$ gives $V^{D,K}(s,x) \le \sup_{\boldsymbol{\nu}\in\mathcal{U}^K} P_D^{\boldsymbol{\nu}}(s,x)$. $\square$

We can use these representations to prove two useful estimates.

Lemma S.2.7. *Let S be a finite $\mathbb{G}$-stopping time, let $\mathcal{H} \subseteq \mathcal{G}$ be a sub-σ-algebra such that S is $\mathcal{H}$-measurable and B^S is a Brownian motion independent of $\mathcal{H}$, and let $X \ge 0$ be $\mathcal{H}$-measurable. Let $\boldsymbol{\pi}$ be a Π_K-valued process that is predictable with respect to $(\mathcal{H} \vee \mathcal{B}_v^S)_{v\ge 0}$, and let Z solve*

$$dZ_v = Z_v(\boldsymbol{\pi}_v^\top \boldsymbol{\theta}\, dv + \boldsymbol{\pi}_v^\top \boldsymbol{\sigma}\, dB_v^S), \qquad Z_0 = X.$$

Then,

(i) for every $A \in \mathcal{H}$ such that $S \le T$ on A,

$$\mathbf{1}_A\, \mathbb{E}\big[F_G(Z_{(T-S)^+}) \mid \mathcal{H}\big] \le \mathbf{1}_A V^{D,K}(S \wedge T, X) \quad a.s.;$$

(ii) if in addition $\sigma_\lambda \sim \mathrm{Exp}(\lambda)$ is independent of $\mathcal{H} \vee \sigma(B^S)$, then

$$\mathbb{E}[F_R(Z_{\sigma_\lambda}) \mid \mathcal{H}] \le V^{R,K}(X) \quad a.s.$$

Proof. Let $\mathbb{P}^\omega := \mathbb{P}(\cdot \mid \mathcal{H})(\omega)$ be a regular conditional probability given $\mathcal{H}$ [Kallenberg, 2021, Thm. 8.5].

Exactly as in the proof of Lemma S.2.6, for $\mathbb{P}$-a.e. ω: under $\mathbb{P}^\omega$ the process B^S is a Brownian motion, every $\mathcal{H}$-set is $\mathbb{P}^\omega$-trivial, $S(\omega)$ and $X(\omega)$ equal constants s and x, and $\boldsymbol{\pi}$ is equal to $\boldsymbol{\nu}^\omega(B^S)$ for $dv \otimes d\mathbb{P}^\omega$-a.e. (v, ω'), for some $\boldsymbol{\nu}^\omega \in \mathcal{U}^K$. Since two controls are equal $dv \otimes d\mathbb{P}^\omega$-a.e. and bounded, their drift terms and Itô integrals once again coincide under $\mathbb{P}^\omega$. Pathwise uniqueness for the SDE gives $Z = X^{x;\boldsymbol{\nu}^\omega}(B^S)$ under $\mathbb{P}^\omega$. If $\omega \in A$, then $s = S(\omega) \le T$, so $(T-S)^+ = T - s$, and Lemma S.2.6 gives

$$\mathbb{E}^{\mathbb{P}^\omega}\left[F_G(Z_{(T-S)^+})\right] = P_D^{\boldsymbol{\nu}^\omega}(s,x) \le V^{D,K}(s,x).$$

Since $A \in \mathcal{H}$, this proves part (i) after multiplying by $\mathbf{1}_A$. For (ii), σ_λ is independent of $\mathcal{H} \vee \sigma(B^S)$ implies that under $\mathbb{P}^\omega$ the variable σ_λ is $\mathrm{Exp}(\lambda)$ and independent of B^S, whence $\mathbb{E}^{\mathbb{P}^\omega}[F_R(Z_{\sigma_\lambda})] = P_R^{\boldsymbol{\nu}^\omega}(x) \le V^{R,K}(x)$ by Lemma S.2.6. $\square$

S.2.2.4 Finite Patch Approximation and Predictable Pasting of Controls

Our last auxiliary results builds a finite approximation of optimal controls.

Lemma S.2.8 (Finite-patch approximation)**.** *For every $\varepsilon > 0$:*

(i) there exist a finite Borel partition $A_1, \dots, A_N$ of $[0,T] \times [0, b_G]$ and controls $\boldsymbol{\nu}^1, \dots, \boldsymbol{\nu}^N \in \mathcal{U}^K$ such that $P_D^{\boldsymbol{\nu}^i}(s,x) \ge V^{D,K}(s,x) - \varepsilon$ whenever $(s,x) \in A_i$;

(ii) there exist a finite Borel partition $B_1, \dots, B_M$ of $[0, b_R]$ and controls $\boldsymbol{\zeta}^1, \dots, \boldsymbol{\zeta}^M \in \mathcal{U}^K$ such that $P_R^{\boldsymbol{\zeta}^j}(y) \ge V^{R,K}(y) - \varepsilon$ whenever $y \in B_j$.

For $x \ge b_G$ in (a) *or $y \ge b_R$ in* (b)*, the zero control is exactly optimal, attaining the value* 1.

Proof. We first record two continuity facts. By scaling, the solution of (S.4) satisfies $X_v^{x;\boldsymbol{\nu}}(\beta) = x\, \mathcal{E}_v^{\boldsymbol{\nu}}$ with $\mathcal{E}_v^{\boldsymbol{\nu}} := \exp\big(\int_0^v (\boldsymbol{\nu}_r^\top \boldsymbol{\theta} - \frac{1}{2}\boldsymbol{\nu}_r^\top \boldsymbol{\Sigma} \boldsymbol{\nu}_r)\, dr + \int_0^v \boldsymbol{\nu}_r^\top \boldsymbol{\sigma}\, d\beta_r\big)$, which has continuous paths and does not depend on x. Hence, for fixed $\boldsymbol{\nu} \in \mathcal{U}^K$, $(s,x) \mapsto x\, \mathcal{E}_{T-s}^{\boldsymbol{\nu}}$ is a.s. jointly continuous, and since $F_G \in [0,1]$ is continuous, bounded convergence gives that $(s,x) \mapsto P_D^{\boldsymbol{\nu}}(s,x) = \mathbb{E}_{\mathbb{W}}[F_G(x\, \mathcal{E}_{T-s}^{\boldsymbol{\nu}})]$ is continuous on $[0,T] \times [0,\infty)$. Likewise, for fixed $\boldsymbol{\zeta} \in \mathcal{U}^K$ and $x \to x'$, $F_R(x\, \mathcal{E}_u^{\boldsymbol{\zeta}}) \to F_R(x'\, \mathcal{E}_u^{\boldsymbol{\zeta}})$ for every (u, ω) by continuity of F_R. Since $F_R \in [0,1]$ dominated convergence on $\lambda e^{-\lambda u}\, du \otimes d\mathbb{W}$ shows $x \mapsto P_R^{\boldsymbol{\zeta}}(x)$ is continuous on $[0, \infty)$.

For (a): fix $\varepsilon > 0$. For each $(s,x) \in [0,T] \times [0, b_G]$, Lemma S.2.6 provides $\boldsymbol{\nu}^{s,x} \in \mathcal{U}^K$ with $P_D^{\boldsymbol{\nu}^{s,x}}(s,x) \ge V^{D,K}(s,x) - \varepsilon/3$. The function $(s',x') \mapsto V^{D,K}(s',x') - P_D^{\boldsymbol{\nu}^{s,x}}(s',x')$ is continuous and is $\le \varepsilon/3$ at (s,x), so it is $< \varepsilon$ on a relatively open neighborhood $U_{s,x}$ of (s,x). Therefore, on $U_{s,x}$, $P_D^{\boldsymbol{\nu}^{s,x}} \ge V^{D,K} - \varepsilon$. These neighborhoods cover the compact set $[0,T] \times [0, b_G]$ and so we can extract a finite subcover $U_{s_1,x_1}, \dots, U_{s_N,x_N}$. From this we can construct a disjoint cover so that we arrive at a Borel partition $A_1, \dots, A_N$ with $A_i \subseteq U_{s_i,x_i}$ and controls $\boldsymbol{\nu}^i := \boldsymbol{\nu}^{s_i,x_i}$. Part (b) is identical on the compact interval $[0, b_R]$. Finally, under the zero control wealth is constant, so if $x \ge b_G$ then $F_G(x) = 1 = V^{D,K}(s,x)$ and if $y \ge b_R$ then $F_R(y) = 1 = V^{R,K}(y)$. $\square$

Remark S.2.9. *In the main proof of Proposition 5.1 we will use the following standard predictable-pasting fact (which follows from a monotone class argument) without proof. Let S be a $\mathbb{G}$-stopping time, let*

$C_1,\dots,C_m$ be a finite $\mathcal{G}_S$-measurable partition, and let $\boldsymbol{\nu}^1,\dots,\boldsymbol{\nu}^m \in \mathcal{U}^K$. Then the process

$$\Pi_r := \sum_{k=1}^{m} \mathbf{1}_{C_k}\, \boldsymbol{\nu}^k_{(r-S)^+}(B^S)\, \mathbf{1}_{\{S<r\}}, \qquad r \geq 0,$$

is $\mathbb{G}$-predictable and Π_K-valued. More generally, finite concatenations of such shifted controls at $\mathbb{G}$-stopping times remain admissible.

S.2.3 Main Proof

We are now ready to complete the proof of the proposition.

Proof of Proposition 5.1. Fix $w \geq 0$ and condition on $\{\tau > t\}$. Write $P := \alpha_R I_R + \alpha_D I_D$, where $I_R := \mathbf{1}_{\{\widehat{W}^{t,w;\boldsymbol{\pi}}_{\tau_t-} \geq R\}}$ and $I_D := \mathbf{1}_{\{\widehat{W}^{t,w;\boldsymbol{\pi}}_{T-} \geq G\}}$. Since $\mathbb{P}(\tau_t = T \mid \tau > t) = 0$,

$$P = P\,\mathbf{1}_{\{\tau_t<T\}} + P\,\mathbf{1}_{\{\tau_t>T\}} \qquad \text{a.s.}$$

Step 1: *(upper bound)* Fix an arbitrary $\boldsymbol{\pi} \in \mathcal{A}^K_{t,\infty}$. By Lemma S.2.2 there is $\boldsymbol{\pi}^0 \in \mathcal{A}^{K,\mathrm{pre}}_{t,T}$ such that

$$\boldsymbol{\pi}_s = \boldsymbol{\pi}^0_s \quad \text{for } ds \otimes d\mathbb{P}\text{-a.e. } (s,\omega) \text{ with } t \leq s < \tau_t \wedge T.$$

Likewise, writing $\widehat{W} = \widehat{W}^{t,w;\boldsymbol{\pi}}$ and $\widetilde{W} = \widetilde{W}^{t,w;\boldsymbol{\pi}^0}$, then almost surely $\widehat{W}_s = \widetilde{W}_s$ for $s \in [t, \tau_t \wedge T)$, and

$$\widehat{W}_{\tau_t-} = \widetilde{W}_{\tau_t} \quad \text{on } \{\tau_t < T\}, \qquad \widehat{W}_{T-} = \widetilde{W}_T \quad \text{on } \{\tau > T\}.$$

Case (i): $(\tau_t < T)$ On $\{\tau_t < T\}$ we have $\widehat{W}_{\tau_t-} = \widetilde{W}_{\tau_t}$, hence $I_R\mathbf{1}_{\{\tau_t<T\}} = \mathbf{1}_{\{\widetilde{W}_{\tau_t}\geq R\}}\mathbf{1}_{\{\tau_t<T\}}$, and the post-payment wealth is $\Gamma_R(\widetilde{W}_{\tau_t})$, which is $\mathcal{G}_{\tau_t}$-measurable. The reward $\alpha_R I_R \mathbf{1}_{\{\tau_t<T\}}$ is also $\mathcal{G}_{\tau_t}$-measurable. For $\alpha_D I_D$: after τ_t the continuation control $\boldsymbol{\pi}|_{[\tau_t,T]}$ is $\mathbb{G}$-predictable. On the stochastic interval $[\![\tau_t, T[\![$, no further non-Brownian information is revealed. More precisely, for each $v \geq 0$,

$$\mathcal{G}_{\tau_t+v}|_{\{\tau_t+v<T\}} = \big(\mathcal{G}_{\tau_t} \vee \mathcal{B}^{\tau_t}_v\big)|_{\{\tau_t+v<T\}}, \qquad \mathcal{B}^{\tau_t}_v := \sigma(B_{\tau_t+r} - B_{\tau_t} : 0 \leq r \leq v),$$

up to completion. A standard argument gives that the shifted control

$$\boldsymbol{\pi}^{\tau_t}_v := \boldsymbol{\pi}_{\tau_t+v}\mathbf{1}_{\{0<v<T-\tau_t\}}, \qquad v \geq 0,$$

is predictable with respect to $(\mathcal{G}_{\tau_t} \vee \mathcal{B}^{\tau_t}_v)_{v\geq 0}$. Consequently, on $\{\tau_t < T\}$, the continuation wealth $\widehat{W}_{T-}$ is $\mathcal{F}_T \vee \sigma(\tau_t, R)$-measurable.

Since G is independent of $\mathcal{F}_\infty \vee \sigma(\tau, R)$, $\mathbb{E}[I_D \mid \mathcal{F}_T \vee \sigma(\tau, R)] = F_G(\widehat{W}_{T-})$, and Lemma S.2.7(i). gives

$$\mathbb{E}[I_D \mid \mathcal{G}_{\tau_t}]\,\mathbf{1}_{\{\tau_t<T\}} = \mathbb{E}\Big[F_G(\widehat{W}_{T-}) \mid \mathcal{G}_{\tau_t}\Big]\,\mathbf{1}_{\{\tau_t<T\}} \leq V^{D,K}\Big(\tau_t, \Gamma_R(\widetilde{W}_{\tau_t})\Big)\,\mathbf{1}_{\{\tau_t<T\}}.$$

Therefore

$$\mathbb{E}\big[P\,\mathbf{1}_{\{\tau_t<T\}}\,\big|\,\mathcal{G}_{\tau_t}\big] \le \Big(\alpha_R \mathbf{1}_{\{\widetilde{W}_{\tau_t}\ge R\}} + \alpha_D V^{D,K}(\tau_t, \Gamma_R(\widetilde{W}_{\tau_t}))\Big)\mathbf{1}_{\{\tau_t<T\}}.$$

Take $\mathbb{E}_{t,w}$ and condition the right-hand side on $\mathcal{F}_\infty \vee \sigma(\tau_t)$; since R is independent of $\mathcal{F}_\infty \vee \sigma(\tau_t)$ and $\widetilde{W}_{\tau_t}$ is $\mathcal{F}_\infty \vee \sigma(\tau_t)$-measurable, the conditional average over R is $\mathcal{J}^K(\tau_t, \widetilde{W}_{\tau_t})$ by (S.6). Lemma S.2.4 then yields

$$\mathbb{E}_{t,w}\big[P\,\mathbf{1}_{\{\tau_t<T\}}\big] \le \int_t^T \lambda e^{-\lambda(s-t)}\,\mathbb{E}_{t,w}\Big[\mathcal{J}^K(s,\widetilde{W}_s)\Big]\,ds. \tag{S.11}$$

<u>*Case (ii):*</u> *($\tau_t > T$)* On $\{\tau_t > T\}$ we have $\widehat{W}_{T-} = \widetilde{W}_T$, hence $I_D\mathbf{1}_{\{\tau_t>T\}} = \mathbf{1}_{\{\widetilde{W}_T\ge G\}}\mathbf{1}_{\{\tau_t>T\}}$ and the post-payment wealth is $\Gamma_G(\widetilde{W}_T)$, which is $\mathcal{K}_T := \mathcal{F}_T \vee \sigma(G)$-measurable. For the random reward, apply Lemma S.2.3 to $\boldsymbol{\pi}|_{[T,\infty)}$: there is an $\mathbb{F}^G$-predictable $\boldsymbol{\pi}^1$ equal to $\boldsymbol{\pi}$ for $ds \otimes d\mathbb{P}$-a.e. (s,ω) with $T \le s < \tau$. Let Z be the wealth process without any payments on $[T,\infty)$ started from $\Gamma_G(\widetilde{W}_T)$ and driven by $\boldsymbol{\pi}^1$ and B^T, i.e.

$$dZ_v = Z_v\big((\boldsymbol{\pi}^1_{T+v})^\top \boldsymbol{\theta}\,dv + (\boldsymbol{\pi}^1_{T+v})^\top \boldsymbol{\sigma}\,dB^T_v\big), \qquad Z_0 = \Gamma_G(\widetilde{W}_T).$$

On $\{\tau_t > T\}$, the actual post-T wealth before the random deadline coincides with Z up to time $\tau_t - T$, so $I_R = \mathbf{1}_{\{Z_{\tau_t-T}\ge R\}}$. Since Z_v is $\mathcal{F}_{T+v}\vee\sigma(G)$-measurable, the process $\psi(s) := \mathbf{1}_{\{Z_{s-T}\ge R\}}$ is bounded, jointly measurable, and $\sigma(\mathcal{F}_s, G, R)$-measurable for every $s \ge T$. By Lemma S.2.5,

$$\mathbb{E}\big[\mathbf{1}_{\{\tau_t>T\}}I_R\,\big|\,\mathcal{K}_T\big] = \mathbb{E}\big[\mathbf{1}_{\{\tau_t>T\}}\psi(\tau_t)\,\big|\,\mathcal{K}_T\big] = e^{-\lambda(T-t)}\,\mathbb{E}\Big[\mathbf{1}_{\{Z_{\sigma_\lambda}\ge R\}}\,\Big|\,\mathcal{K}_T\Big],$$

with $\sigma_\lambda \sim \mathrm{Exp}(\lambda)$ independent of $\mathcal{F}_\infty \vee \sigma(R,G)$. Since R is independent of $\mathcal{K}_T \vee \sigma(B^T) \vee \sigma(\sigma_\lambda)$, the last expectation is $\mathbb{E}[F_R(Z_{\sigma_\lambda}) \mid \mathcal{K}_T]$, and Lemma S.2.7(ii) gives $\mathbb{E}[F_R(Z_{\sigma_\lambda}) \mid \mathcal{K}_T] \le V^{R,K}(\Gamma_G(\widetilde{W}_T))$. Hence

$$\mathbb{E}\big[\alpha_R I_R\,\mathbf{1}_{\{\tau_t>T\}}\,\big|\,\mathcal{K}_T\big] \le \alpha_R e^{-\lambda(T-t)}\,V^{R,K}\Big(\Gamma_G(\widetilde{W}_T)\Big).$$

For the fixed reward, $\mathbf{1}_{\{\widetilde{W}_T\ge G\}}$ is $\mathcal{K}_T$-measurable and τ_t is independent of $\mathcal{K}_T$ under $\mathbb{P}(\cdot \mid \tau > t)$, so $\mathbb{E}[\alpha_D I_D \mathbf{1}_{\{\tau_t>T\}} \mid \mathcal{K}_T] = \alpha_D e^{-\lambda(T-t)}\mathbf{1}_{\{\widetilde{W}_T\ge G\}}$. Adding, taking $\mathbb{E}_{t,w}$, and averaging over G (conditioning on $\mathcal{F}_\infty$, since G is independent of $\mathcal{F}_\infty$) we obtain (by (S.7)),

$$\mathbb{E}_{t,w}\big[P\,\mathbf{1}_{\{\tau_t>T\}}\big] \le e^{-\lambda(T-t)}\,\mathbb{E}_{t,w}\Big[\mathcal{T}^K(\widetilde{W}_T)\Big]. \tag{S.12}$$

Adding (S.11) and (S.12) bounds the payoff of $\boldsymbol{\pi}$ by the functional in (S.10) evaluated at $\boldsymbol{\pi}^0 \in \mathcal{A}^{K,\mathrm{pre}}_{t,T}$. Taking the supremum over $\boldsymbol{\pi} \in \mathcal{A}^K_{t,\infty}$ gives $V^{R,D,K}(t,w) \le \widetilde{V}^{R,D,K}(t,w)$.

<u>***Step 2:***</u> *(lower bound)* Fix $\boldsymbol{\pi} \in \mathcal{A}^{K,\mathrm{pre}}_{t,T}$, write $\widetilde{W} = \widetilde{W}^{t,w;\boldsymbol{\pi}}$, fix $\varepsilon > 0$, and take the finite patches of Lemma S.2.8. Define a full strategy $\boldsymbol{\pi}^\varepsilon$ as follows. First, use $\boldsymbol{\pi}$ on $[t, \tau_t \wedge T)$. Then, on $\{\tau_t = s < T\}$ set $X_s := \Gamma_R(\widetilde{W}_s)$ and, on $[s,T]$, use the zero control if $X_s \ge b_G$ and otherwise use the shifted control $\boldsymbol{\nu}^i(B^{\tau_t})$ when $(s, X_s) \in A_i$. After T on this branch set the control equal to zero. On $\{\tau_t > T\}$ set $Y_T := \Gamma_G(\widetilde{W}_T)$ and, on $[T,\tau_t)$, use the zero control if $Y_T \ge b_R$ and otherwise use the shifted control $\boldsymbol{\zeta}^j(B^T)$ when $Y_T \in B_j$. After τ_t set the control equal to zero. The switching events are $\mathcal{G}_{\tau_t}$- and $\mathcal{G}_T$-measurable and occur at the $\mathbb{G}$-stopping times τ_t and T. By Remark S.2.9, $\boldsymbol{\pi}^\varepsilon \in \mathcal{A}^K_{t,\infty}$, and its pre-payment wealth coincides with $\widetilde{W}$ on

$[t, \tau_t \wedge T)$.

Case (i): ($\tau_t < T$) On $\{\tau_t = s < T\}$ the continuation wealth at T is $\widehat{W}_{T-} = X_{T-s}^{X_s;\boldsymbol{\nu}^i}(B^{\tau_t})$ on $\{(s, X_s) \in A_i\}$ (resp. X_s under the zero control). Because G is independent of $\mathcal{F}_\infty \vee \sigma(\tau, R)$, $\boldsymbol{\nu}^i$ is a fixed functional of B^{τ_t}, B^{τ_t} is independent of $\mathcal{G}_{\tau_t}$, and (s, X_s) is $\mathcal{G}_{\tau_t}$-measurable, freezing the $\mathcal{G}_{\tau_t}$-measurable parameters and integrating the independent increments gives

$$\mathbb{E}[I_D \mid \mathcal{G}_{\tau_t}]\, \mathbf{1}_{\{(s,X_s)\in A_i\}} = P_D^{\boldsymbol{\nu}^i}(s, X_s)\, \mathbf{1}_{\{(s,X_s)\in A_i\}} \geq \big(V^{D,K}(s, X_s) - \varepsilon\big)\mathbf{1}_{\{(s,X_s)\in A_i\}},$$

while on $\{X_s \geq b_G\}$, $\mathbb{E}[I_D \mid \mathcal{G}_{\tau_t}] = 1 = V^{D,K}(s, X_s)$. In all cases

$$\mathbb{E}[I_D \mid \mathcal{G}_{\tau_t}]\mathbf{1}_{\{\tau_t<T\}} \geq (V^{D,K}(\tau_t, \Gamma_R(\widetilde{W}_{\tau_t})) - \varepsilon)\mathbf{1}_{\{\tau_t<T\}}.$$

Proceeding as in Step 1 (conditioning on $\mathcal{F}_\infty \vee \sigma(\tau_t)$ to average over R, then Lemma S.2.4),

$$\mathbb{E}_{t,w}\big[P\, \mathbf{1}_{\{\tau_t<T\}}\big] \geq \int_t^T \lambda e^{-\lambda(s-t)}\, \mathbb{E}_{t,w}\Big[\mathcal{J}^K(s, \widetilde{W}_s) - \alpha_D\varepsilon\Big]\, ds.$$

Case (ii): ($\tau_t > T$) On $\{\tau_t > T\}$ the control $\boldsymbol{\zeta}^j$ (resp. the zero control) is $\mathbb{B}$-predictable, so the continuation wealth $Z_v = X_v^{Y_T;\boldsymbol{\zeta}^j}(B^T)$ is $\mathbb{F}^G$-adapted and $\psi(s) := \mathbf{1}_{\{Z_{s-T}\geq R\}}$ is $\sigma(\mathcal{F}_s, G, R)$-measurable. By Lemma S.2.5, the independence of R from $\mathcal{K}_T \vee \sigma(B^T) \vee \sigma(\sigma_\lambda)$, and an application of the same argument on $\{Y_T \in B_j\}$ where the $\mathcal{K}_T$-measurable values are frozen and the independent randomness is integrated out,

$$\mathbb{E}\big[\mathbf{1}_{\{\tau_t>T\}} I_R \,\big|\, \mathcal{K}_T\big] = e^{-\lambda(T-t)}\mathbb{E}[F_R(Z_{\sigma_\lambda}) \mid \mathcal{K}_T] = e^{-\lambda(T-t)} P_R^{\boldsymbol{\zeta}^j}(Y_T) \geq e^{-\lambda(T-t)}\big(V^{R,K}(Y_T) - \varepsilon\big),$$

while on $\{Y_T \geq b_R\}$, $\mathbb{E}[F_R(Z_{\sigma_\lambda}) \mid \mathcal{K}_T] = F_R(Y_T) = 1 = V^{R,K}(Y_T)$. The fixed reward contributes $\alpha_D e^{-\lambda(T-t)}\mathbf{1}_{\{\widetilde{W}_T\geq G\}}$ as in Step 1. Adding, taking $\mathbb{E}_{t,w}$ and averaging over G,

$$\mathbb{E}_{t,w}\big[P\, \mathbf{1}_{\{\tau_t>T\}}\big] \geq e^{-\lambda(T-t)}\, \mathbb{E}_{t,w}\Big[\mathcal{T}^K(\widetilde{W}_T) - \alpha_R\varepsilon\Big].$$

Adding the two estimates and writing

$$C_{t,T} := \alpha_D \int_t^T \lambda e^{-\lambda(s-t)}\, ds + \alpha_R e^{-\lambda(T-t)} = \alpha_D + (\alpha_R - \alpha_D)e^{-\lambda(T-t)} \leq \alpha_D + \alpha_R = 1,$$

we obtain

$$V^{R,D,K}(t, w) \geq \mathbb{E}_{t,w}\Big[\int_t^T \lambda e^{-\lambda(s-t)} \mathcal{J}^K(s, \widetilde{W}_s)\, ds + e^{-\lambda(T-t)}\mathcal{T}^K(\widetilde{W}_T)\Big] - \varepsilon\, C_{t,T},$$

so the functional in (S.10) evaluated at $\boldsymbol{\pi}$ is at most $V^{R,D,K}(t, w) + \varepsilon$. Letting $\varepsilon \downarrow 0$ and taking the supremum over $\boldsymbol{\pi} \in \mathcal{A}_{t,T}^{K,\text{pre}}$ gives $\widetilde{V}^{R,D,K}(t, w) \leq V^{R,D,K}(t, w)$. Taking this together with Step 1 completes the proof of Proposition 5.1. $\square$

S.3 Sensitivity to Calibration Parameters

S.3.1 Forced Funding Sensitivity Analysis

We show that the qualitative findings of the growth crowding-out effect and the deadline pressure effect from Section 6.3 are robust to variations in goal dispersion, goal amount specification, college saving horizons, and preference weights. Figures S.1 and S.3 vary the dispersion σ_G of the fixed-deadline goal distribution across retirement and college calibrations respectively. Larger σ_G fattens the right tail of G, raising the worst-case goal realization and shifting value curves rightward. The policy response reflects two opposing forces: higher σ_G expands the safe level b_G, keeping π^* elevated over a wider wealth range, but also places more mass on low realizations of G that are achievable even from modest wealth, partially offsetting the downward pressure on value.

Figure S.2 replaces the deterministic random-deadline goal R with a truncated lognormal of the same mean and standard deviation $\$15,499.06$. The value and policy curves are nearly indistinguishable from the deterministic benchmark, showing that the model's main findings are not sensitive to the specification of the random-deadline goal distribution. The only visible difference is a marginally wider hump in the optimal policy in the intermediate wealth region, arising because some lognormal realizations of R exceed the deterministic benchmark and require a slightly wider hedging band.

Figures S.4 and S.5 vary the saving horizon T. The deadline pressure effect documented in Section 6.3 is present here as well: shorter horizons shift value curves rightward and force the household to maintain peak risk-taking over a wider wealth range before spiking sharply near the goal threshold. When the fixed-deadline goal is calibrated to the private college target of \$262,000, the household requires substantially more wealth to achieve any given success probability than under the public college target of \$124,000, but the deadline pressure effect operates identically across both goal amounts.

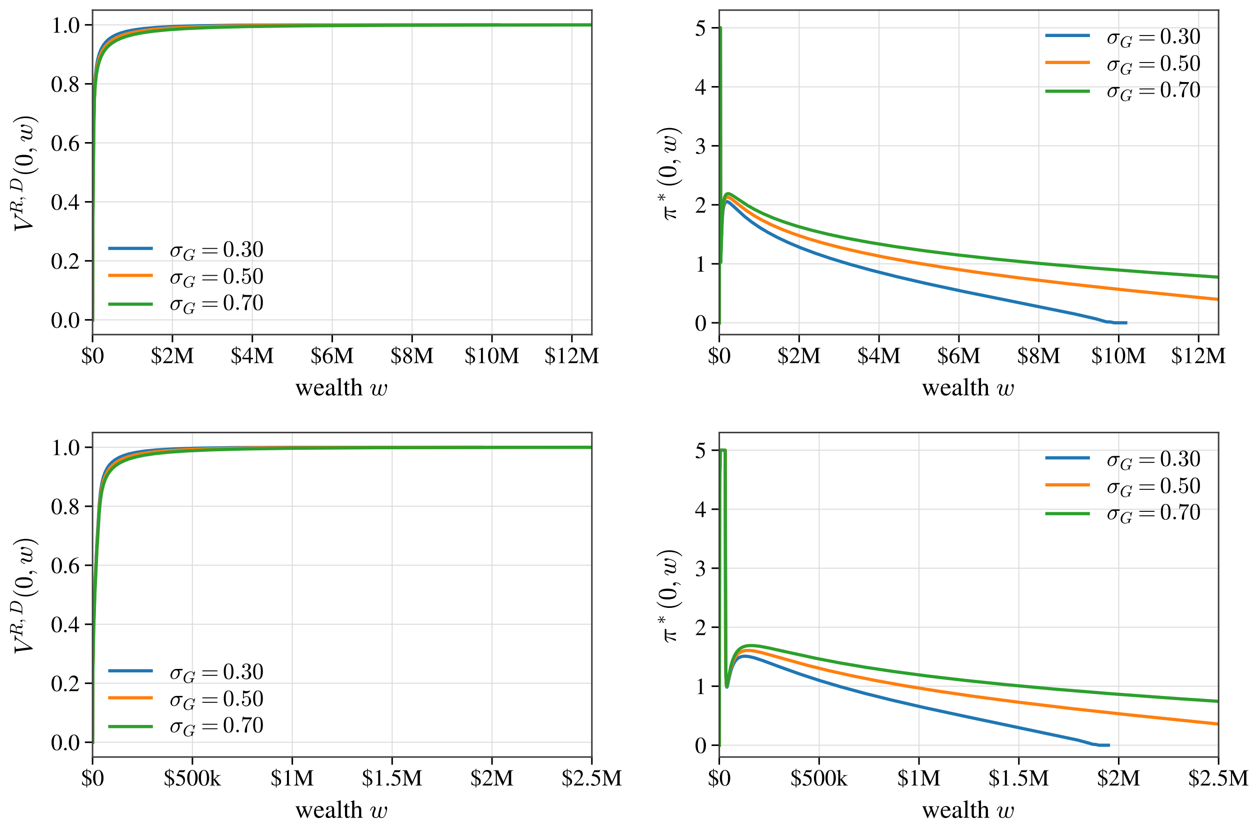


Figure S.1: Sensitivity of the initial two-goal value function $V^{R,D}(0, w)$ and the initial optimal risky position $\pi^*(0, w)$ to the dispersion σ_G of the fixed-deadline goal. The top row uses the high-income retirement target with median goal amount $13.75 \times \$300{,}000 = \$4.125$ million, and the bottom row uses the low-income retirement target with median goal amount $26 \times \$30{,}000 = \$780{,}000$. In all panels, we fix $T = 40$.

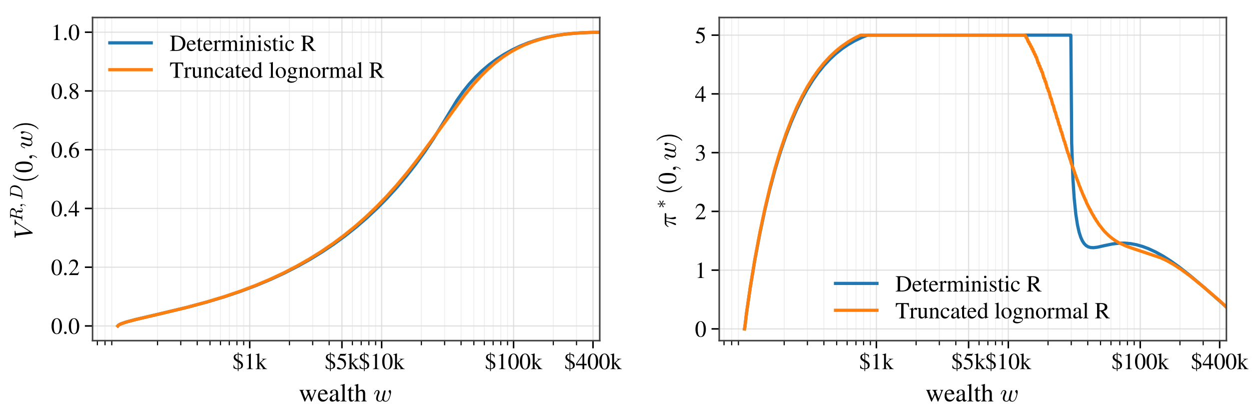


Figure S.2: Sensitivity to the specification of the random-deadline goal for the public-college baseline. R is deterministic at $29,837.40 with $\lambda = 0.20$. The stochastic alternative uses a truncated lognormal distribution with mean $29,837.40 and log-standard-deviation $\sigma_{\log R} = 0.50$. This implies standard deviation $15,499.06. The fixed-deadline goal is a truncated lognormal with public-college median $G = \$124{,}000$, $\sigma_G = 0.50$, and $T = 18$.

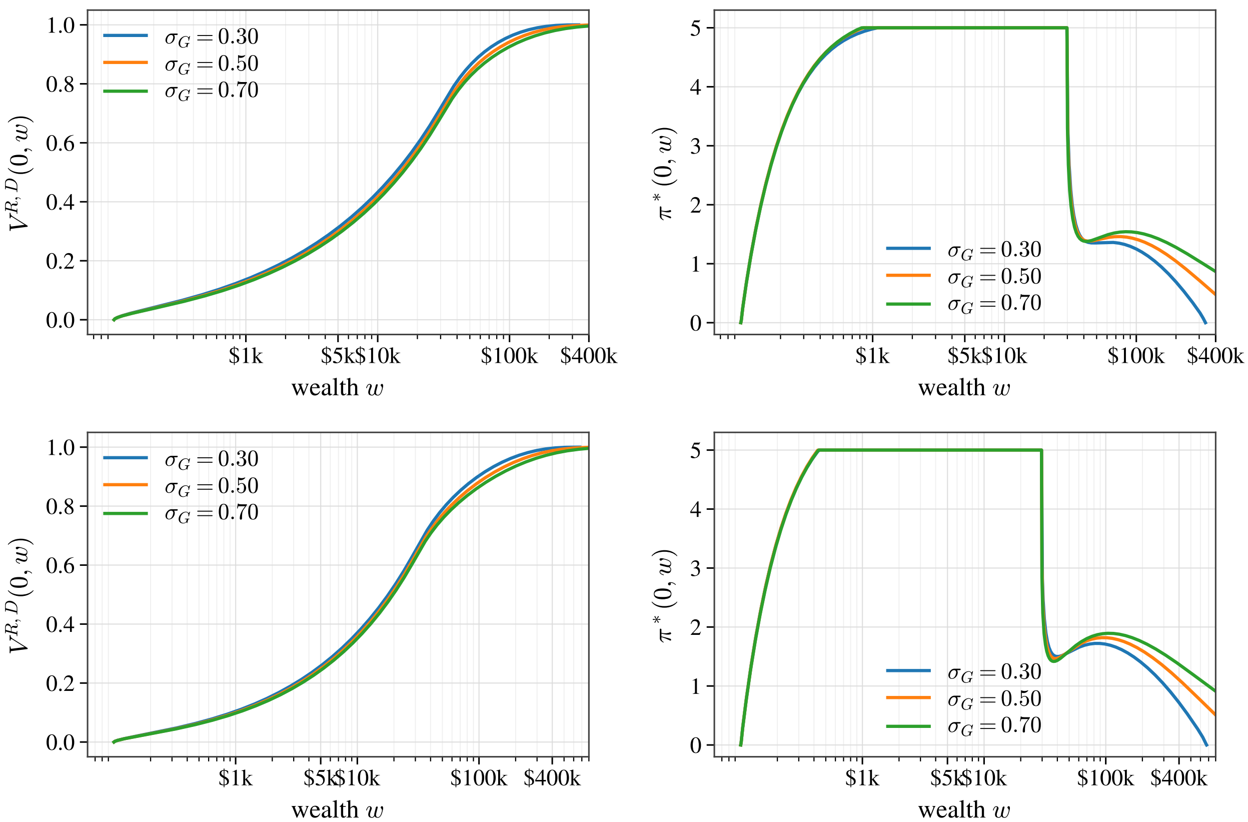


Figure S.3: Sensitivity to the fixed-deadline goal dispersion σ_G for public and private four-year college targets. The top row uses the public-college median goal \$124,000, and the bottom row uses the private-college median goal \$262,000.

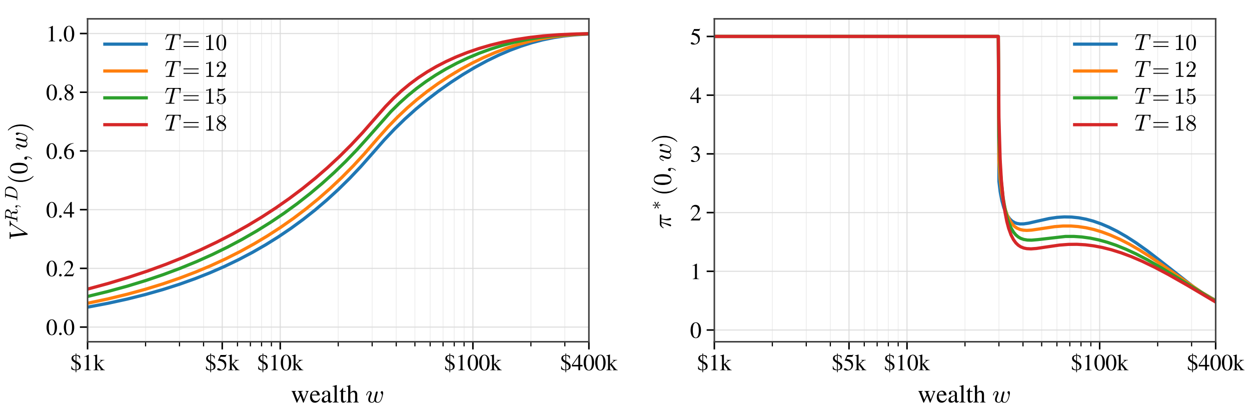


Figure S.4: Sensitivity of the initial two-goal value function $V^{R,D}(0, w)$ and the initial optimal risky position $\pi^*(0, w)$ to the college saving horizon T under the public-college calibration. T varies from 10 to 18 years, with $T = 18$ corresponding to saving from birth and smaller values corresponding to later entry into the saving period.

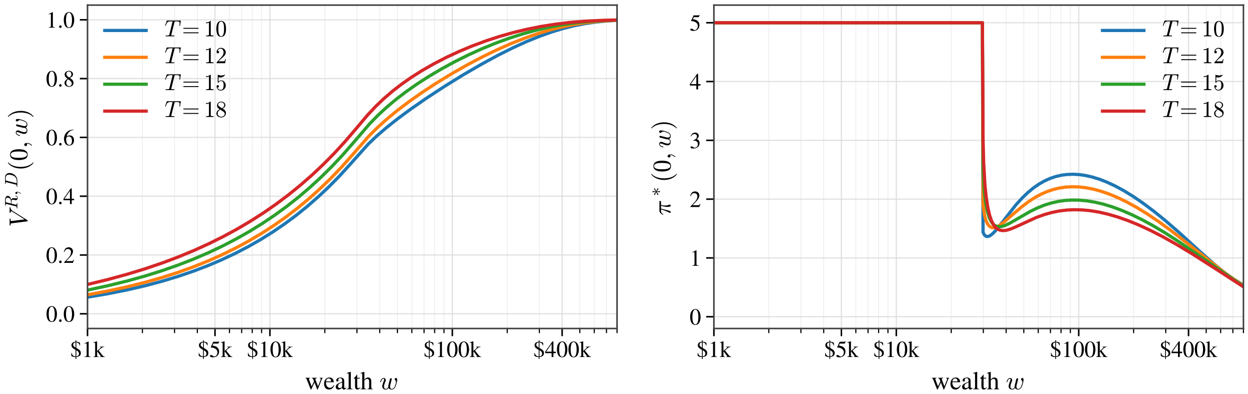
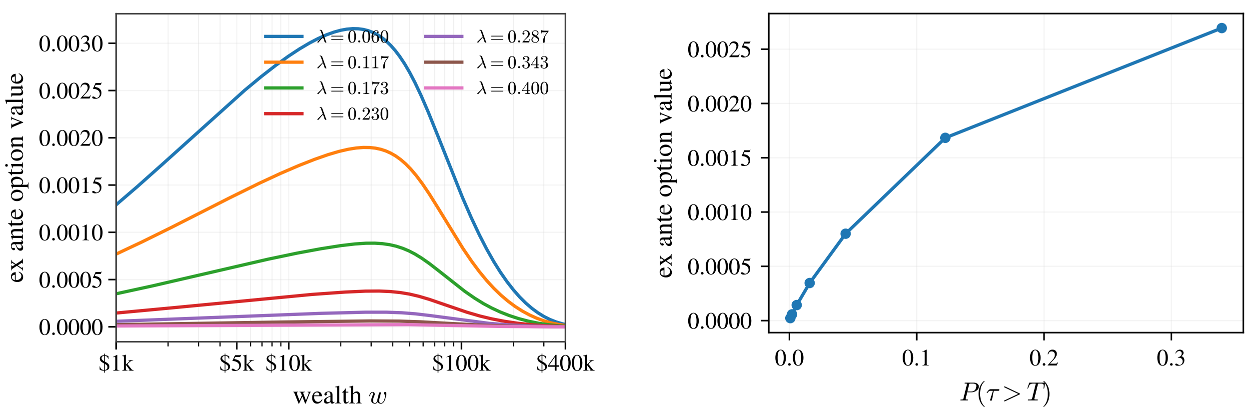


Figure S.5: Sensitivity of the initial two-goal value function $V^{R,D}(0, w)$ and the initial optimal risky position $\pi^*(0, w)$ to the college saving horizon T under the private-college calibration. T varies from 10 to 18 years, with $T = 18$ corresponding to saving from birth and smaller values corresponding to later entry into the saving period.

S.3.2 Robustness of Optional Funding Results

In this section, we complement the optional funding analysis in Section 6.4. We show the sensitivity of the ex ante option value and the terminal option value to the random-deadline arrival intensity λ, the fixed-goal priority per dollar α_D/G, and the deterministic deadline T.

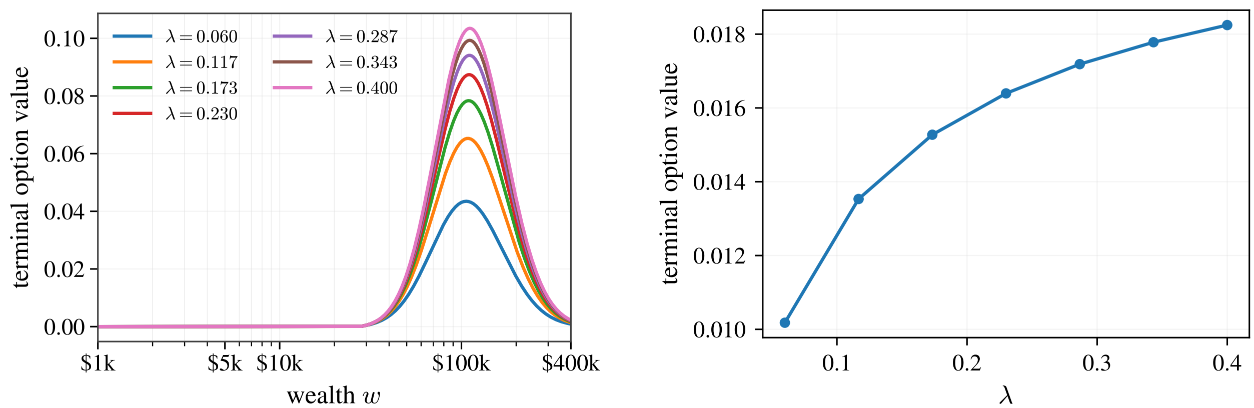
In Figures S.6 and S.7, we show the effect of varying λ. The ex ante option value is larger when λ is smaller because a lower arrival intensity increases the probability $e^{-\lambda T}$ that the random-deadline goal survives past T, which is precisely the branch on which the funding decision is available. The terminal option value, by contrast, isolates the time-T tradeoff directly and is therefore less sensitive to λ through the survival probability channel, though it is still affected through the shape of $V^{R,K}$.



(a) Ex ante option value by wealth.

(b) Ex ante option value versus $\Pr(\tau > T) = e^{-\lambda T}$.

Figure S.6: Sensitivity of the ex ante option value $\Delta V_0(w_0)$ to the random-deadline arrival intensity λ. The right panel plots, at a fixed wealth level $\$50k$, the ex ante option value against the survival probability $\Pr(\tau > T) = e^{-\lambda T}$: the option is more valuable when the random-deadline goal is more likely to remain unresolved at T.

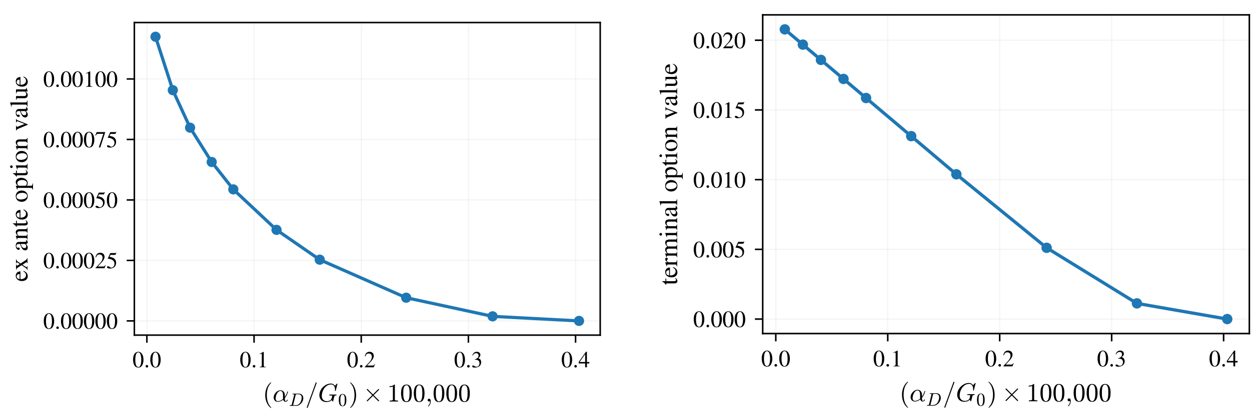


(a) Terminal option value by wealth.

(b) Terminal option value versus λ.

Figure S.7: Sensitivity of the terminal option value $\Delta\mathcal{T}(w)$ to the random-deadline arrival intensity λ at a fixed wealth of $\$50k$. The terminal option value isolates the time-T funding tradeoff and is evaluated conditional on the random-deadline goal remaining unresolved at T.

In Figure S.8, we vary the fixed-goal priority per dollar α_D/G. As this ratio rises, the direct reward from funding the fixed-deadline goal increasingly outweighs the loss in random-deadline continuation value, so the household prefers to fund the goal whenever affordable and the option to decline it loses value. Both the ex ante option value and the terminal option value fall monotonically and eventually vanish once the fixed-goal reward is sufficiently large relative to the continuation value preserved by not funding.

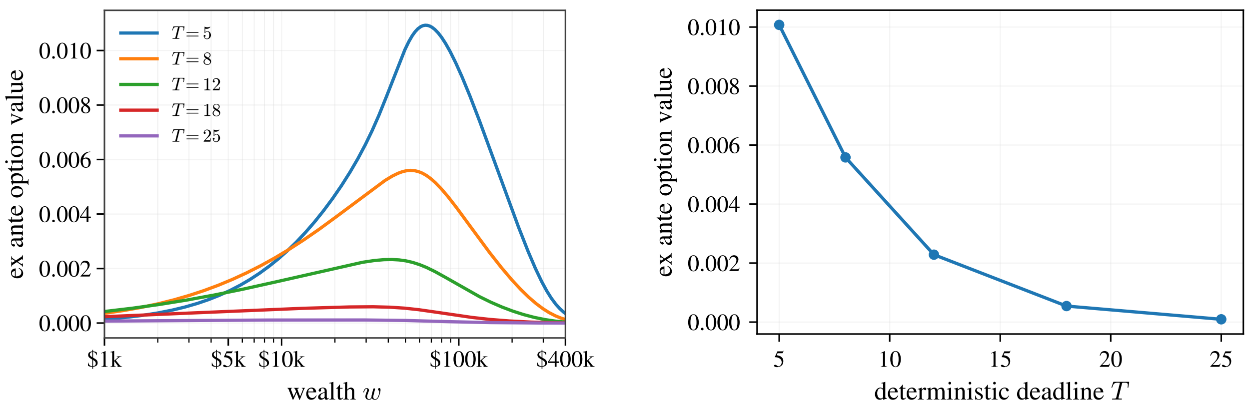


(a) Ex ante option value.

(b) Terminal option value.

Figure S.8: Ex ante option value and terminal option value as a function of the fixed-deadline goal priority per dollar $(\alpha_D/G) \times 100{,}000$. Initial wealth is fixed at $w_0 = \$50k$. Both objects fall monotonically as the fixed-goal reward rises relative to the random-deadline continuation value.

In Figure S.9, we vary the deterministic deadline T. A longer horizon gives the household more time to accumulate wealth before the fixed-deadline goal is processed, but it also reduces the probability that the random-deadline goal survives until T. These two forces work in opposite directions on the ex ante option value: a longer T improves the terminal tradeoff by allowing more wealth accumulation, but discounts it more heavily through the lower survival probability $e^{-\lambda T}$.

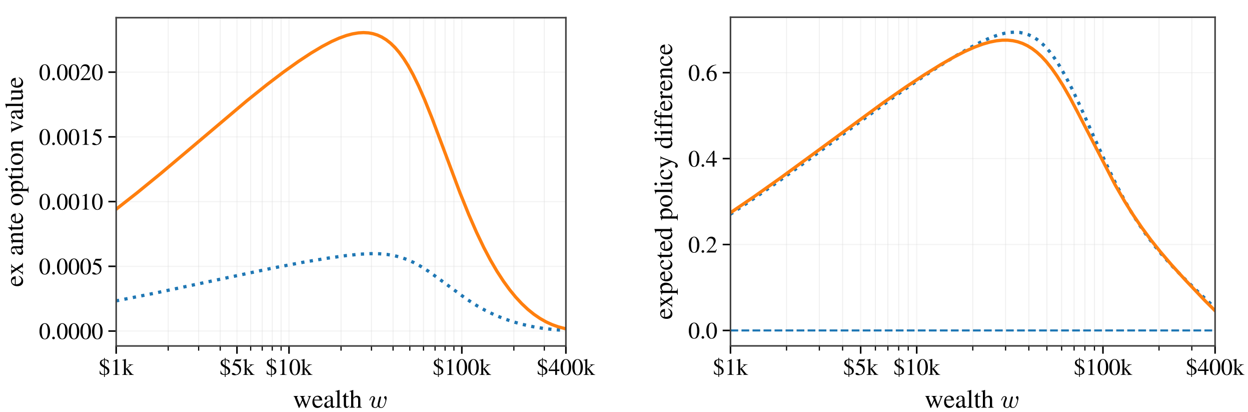


(a) Ex ante option value by wealth.

(b) Ex ante option value versus T at fixed wealth $\$50k$.

Figure S.9: Sensitivity of the ex ante option value $\Delta V_0(w_0)$ to the deterministic deadline T. Longer horizons allow more wealth accumulation before the fixed-deadline funding decision but reduce the survival probability $e^{-\lambda T}$ of the random-deadline goal, producing a non-monotone relationship between T and the ex ante option value.

Figure S.10 shows the same ex ante option value and policy difference under a lower arrival intensity λ. The survival probability $\Pr(\tau > T) = e^{-\lambda T}$ is higher, the option is more valuable, and the policy difference between optional and forced funding is correspondingly larger at intermediate wealth, consistent with the pattern documented in Figure 7.



(a) Ex ante option value.

(b) Optimal policy difference between optional and forced funding.

Figure S.10: Ex ante option value and optimal policy difference under a lower arrival intensity λ, which raises the survival probability $\Pr(\tau > T) = e^{-\lambda T}$. Both the option value and the policy difference are larger than under the baseline λ, consistent with the survival-probability channel documented in Figure S.6.

S.4 Additional Results on Asymptotics

This appendix expands on the limiting cases of our framework discussed in Section 6.5. There are three types of limits. First, we study the role of the deterministic deadline T in the full two-goal problem. Second, we study the limiting regimes generated by the random-arrival intensity λ. Third, we vary the fixed-goal priority per dollar.

S.4.1 Deterministic-Deadline Asymptotics

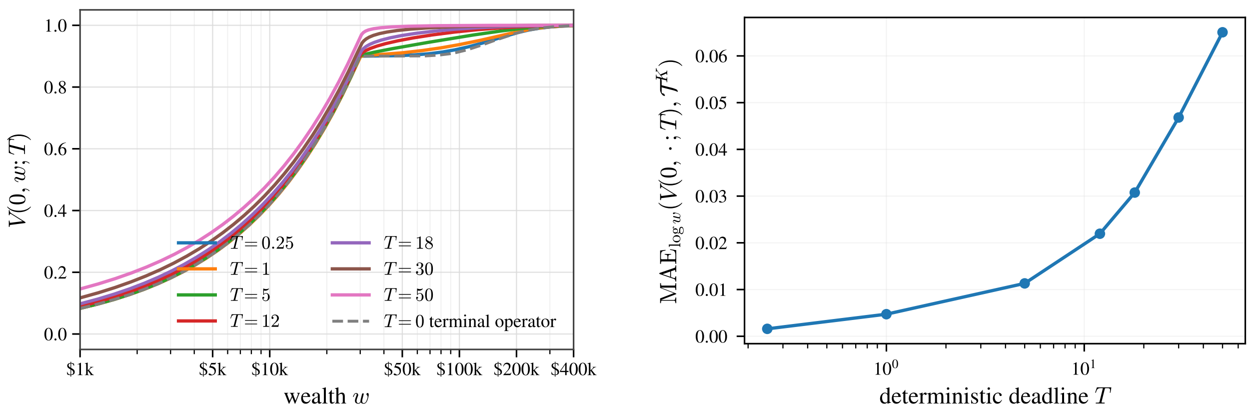
We study the role of the deterministic deadline T in the full two-goal problem. When $T \to 0$, there is essentially no time to rebalance or accumulate wealth before the fixed-deadline goal is processed. Hence the time-0 two-goal value converges to the terminal operator

$$V^{R,D,K}(0, w; T) \longrightarrow \mathcal{T}^K(w), \qquad T \downarrow 0, \tag{S.13}$$

where

$$\mathcal{T}^K(w) = \mathbb{E}_G\left[\alpha_D \mathbf{1}_{\{w \geq G\}} + \alpha_R V^{R,K}\left(w - G\mathbf{1}_{\{w \geq G\}}\right)\right]. \tag{S.14}$$

Thus the $T \to 0$ limit is a single terminal funding decision followed by the random-deadline continuation problem.

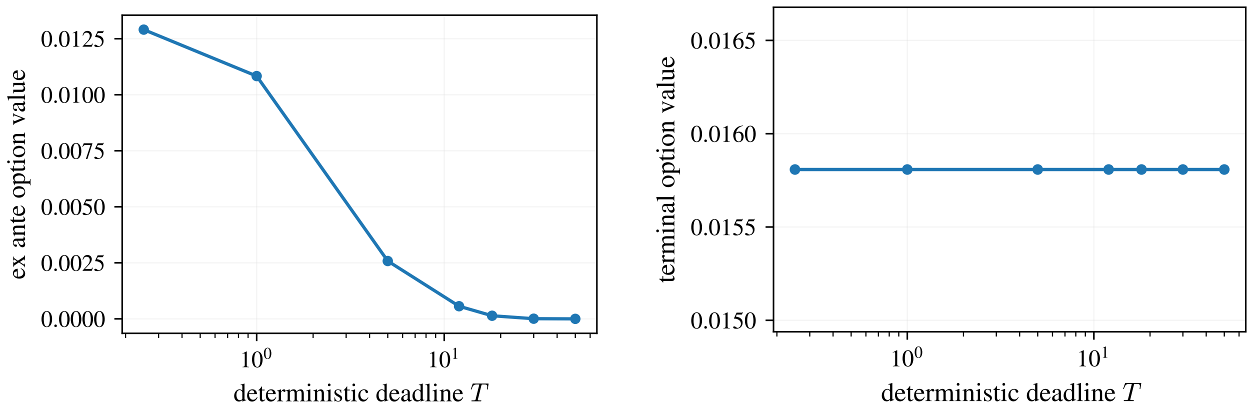


(a) Value function as T varies.

(b) Distance to terminal operator.

Figure S.11: Value function convergence in T. Panel (a) plots the time-0 value function $V(0, w; T)$ for different deterministic deadlines T, together with the terminal operator $\mathcal{T}^K(w)$. Panel (b) plots the average absolute error over log wealth $\mathrm{MAE}_{\log w}(V, \mathcal{T}^K) = \frac{1}{\log w_{\max} - \log w_{\min}} \int_{\log w_{\min}}^{\log w_{\max}} |V(e^x) - \mathcal{T}^K(e^x)|\, dx$. As $T \to 0$, the value converges to the terminal operator because there is no time to trade before the fixed-deadline goal is processed.

The option value also depends on T, but for a different reason. The terminal option to decline the fixed-deadline goal matters only on the branch $\{\tau > T\}$, where the random-deadline goal remains unresolved at the deterministic deadline. Since $\mathbb{P}(\tau > T) = e^{-\lambda T}$, the time-0 value of the option is diluted as T grows. This does not mean that the terminal tradeoff disappears; rather, the branch on which it matters is reached with lower probability.



(a) Ex ante option value.

(b) Terminal option value.

Figure S.12: Option value as the deterministic deadline varies. Panel (a) plots the ex ante option value $\{V_{\text{opt}}(0, w; T) - V_{\text{forced}}(0, w; T)\}$ at a fixed wealth $w = \$50k$. Panel (b) plots the maximum conditional terminal option value, $\max_w\{\mathcal{T}^K_{\text{opt}}(w) - \mathcal{T}^K_{\text{forced}}(w)\}$. The time-0 option value falls for long deadlines because the probability that the random-deadline goal survives until T is $e^{-\lambda T}$, while the conditional terminal object isolates the time-T tradeoff itself.

S.4.2 Random-Arrival Asymptotics

The random-arrival intensity λ generates two useful limiting regimes. When $\lambda \to 0$, the random-deadline goal is unlikely to arrive before the deterministic deadline. In this regime, the wealth dependence of the two-goal value is increasingly governed by the fixed-deadline accumulation problem. For deterministic G, the value has the Browne-shaped approximation

$$V^{R,D,K}(0, w) \approx \alpha_R + \alpha_D V^D_{\text{Browne}}(0, w; G), \qquad \lambda \approx 0. \tag{S.15}$$

The additive α_R term reflects the limiting random-deadline goal continuation component, while the nontrivial wealth dependence comes from the fixed-deadline Browne benchmark.

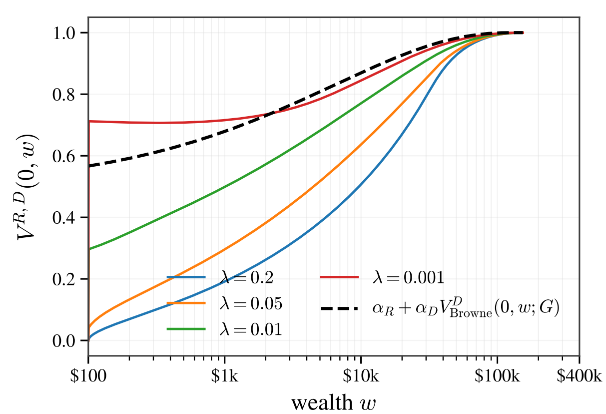
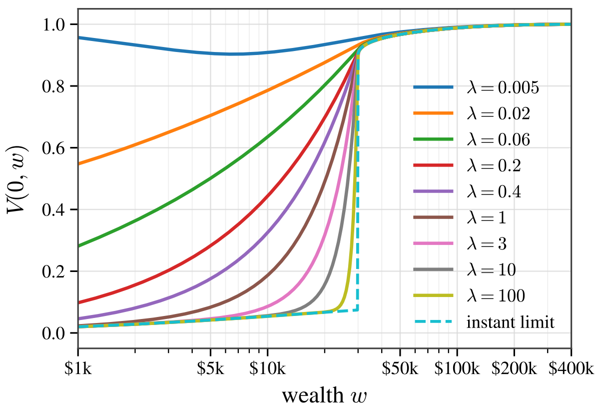


Figure S.13: Small-λ limit. As the random-arrival intensity decreases, the two-goal value becomes increasingly Browne-shaped. The plotted reference is $\alpha_R + \alpha_D V^D_{\text{Browne}}(0, w; G)$.

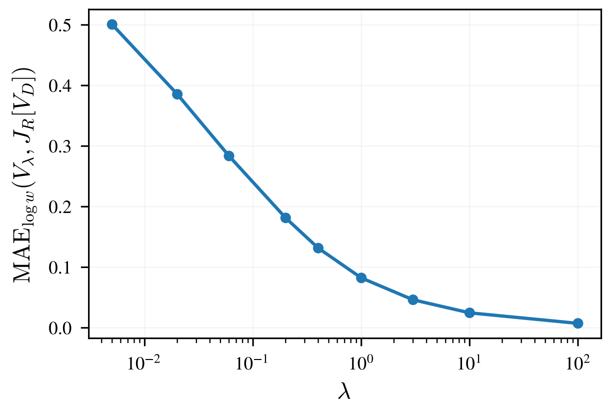
At the opposite extreme, as $\lambda \to \infty$, the random-deadline goal is resolved immediately. The limiting value is therefore the instant-arrival operator applied at time zero:

$$\mathcal{J}_R[V^D](0,w) = \mathbb{E}_R\left[\alpha_R \mathbf{1}_{\{w\geq R\}} + \alpha_D V^{D,K}\left(0, w - R\mathbf{1}_{\{w\geq R\}}\right)\right]. \tag{S.16}$$

This is the same object that appears as the HJB source term. Economically, (S.16) says that the random-deadline goal is processed immediately and the remaining problem is the fixed-deadline single-goal continuation problem.



(a) Value function for different λ.



(b) Distance to instant-arrival reference.

Figure S.14: Random-arrival asymptotics. Panel (a) plots $V(0,w)$ for different arrival intensities together with the instant-arrival reference $\mathcal{J}_R[V^D](0,w)$. Panel (b) reports the average absolute error over log wealth between the computed two-goal value and the instant-arrival reference.

The option value is not monotone in λ. The terminal option matters only if the random-deadline goal survives until T, which favors small λ. However, when λ is too small, the remaining random-deadline goal is not very urgent, so preserving wealth for it is less valuable. The option is therefore largest at intermediate arrival intensities, where the random-deadline goal is likely enough to survive until T but also urgent enough to make the post-T continuation value economically significant.

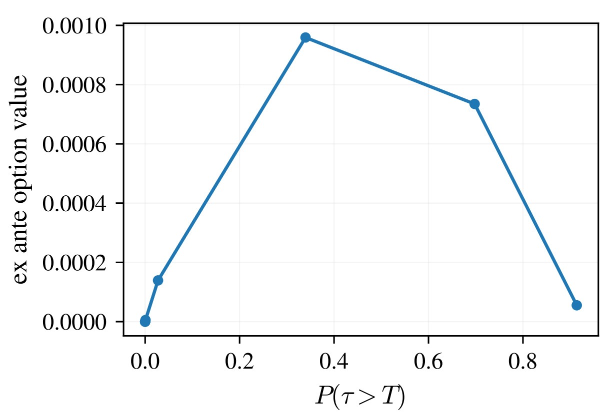


Figure S.15: Ex ante option value as a function of the random-deadline survival probability $\Pr(\tau > T) = e^{-\lambda T}$. Moving to the right corresponds to lowering λ. Wealth is fixed at \$50k.

References

Anna Aksamit and Monique Jeanblanc. *Enlargement of filtration with finance in view*. Springer, 2017.

Michael G Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1):1–67, 1992.

Olav Kallenberg. *Foundations of modern probability*. Springer, third edition, 2021.

Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.

Nizar Touzi. *Optimal stochastic control, stochastic target problems, and backward SDE*, volume 29. Springer Science & Business Media, 2013.